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Uniformly defining p-henselian valuations ☆

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ABSTRACT

Admitting a non-trivial p-henselian valuation is a weaker assumption on a field than admitting a non-trivial henselian valuation. Unlike henselianity, p-henselianity is an elementary property in the language of rings. We are interested in the question when a field admits a non-trivial 0-definable p-henselian valuation (in the language of rings). We give a classification of elementary classes of fields in which the canonical p-henselian valuation is uniformly 0-definable. We then apply this to show that there is a definable valuation inducing the (t-)henselian topology on any (t-)henselian field which is neither separably closed nor real closed.

1. Introduction

Where a valuation v on a field K contributes to the arithmetic of K, e.g., in the sense that the existence of K-rational points on certain algebraic varieties defined over K is guaranteed or prohibited by 'local' conditions 'at v', the valuation v (or rather its valuation ring $\mathcal{O}_v$) is often definable by a first-order formula $\phi(x)$ in the language of rings $\mathcal{L}_{\text{ring}} = \{+, \times; 0, 1\}$: For each $a \in K$, one has $a \in \mathcal{O}_v$ if and only if $\phi(a)$ holds in K – we then write $\mathcal{O}_v = \phi(K)$.

This happens, for example, for all valuations in all global fields (a fact implicit in the pioneering works [10] and [11] of Julia Robinson), and later, Rumely even found a *uniform* first-order definition for all valuation rings in all global fields [12]. It also happens in the classical henselian fields $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ or $k((t))$ for an arbitrary field of coefficients k via the well known formulas for $\mathbb{Z}_p$ in $\mathbb{Q}_p$ and for $k[[t]]$ in $k((t))$ due to Ax

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and others. It does not happen on $\mathbb{C}$ or on $\mathbb{R}$ or on any algebraically or real closed field, where no valuation is of arithmetical interest, and where no non-trivial valuation is first-order definable, because, by quantifier elimination, first-order definable subsets of algebraically closed fields are finite or cofinite and those on real closed fields are finite unions of intervals and points.

In the 1970's the concept of a *2-henselian* valuation emerged from the algebraic theory of quadratic forms, and later, by way of analogy, the notion of a p-henselian valuation was coined for an arbitrary prime number p: A valuation v on a field K is called *p-henselian* if v has a unique prolongation to $K(p)$, the maximal Galois-p extension of K (i.e., the compositum of all finite Galois extensions of p-power degree over K in some fixed algebraic closure of K). Equivalently, v is p-henselian on K if it has a unique prolongation to each Galois extension of degree p – this fact that p-henselianity shows in Galois extensions of bounded degree makes it easier to find definable p-henselian valuations compared to finding definable henselian valuations. Note that every henselian valuation is p-henselian but, in general, not the other way round.

Like for henselian valuations there may be several p-henselian valuations on a field K, but there always is a canonical one: the *canonical p-henselian valuation* v_K^p on a field K is the coarsest p-henselian valuation v on K whose residue field Kv is p-closed (i.e., where $Kv = Kv(p)$) if there is any such; if not it is the finest p-henselian valuation on K (cf. Section 3 of [5] where existence and uniqueness of v_K^p is proven). Recall that, for two valuations v, w on K, v is finer than w just in case $\mathcal{O}_v \subseteq \mathcal{O}_w$. Recall further that if v is finer than w, then, equivalently, w is coarser than v. The valuation v_K^p is non-trivial if and only if K admits a non-trivial p-henselian valuation.

This paper is intended to both close a gap in the proof of Theorem 3.2 of [5] and to present a more uniform version of the Theorem. This Theorem asserts that v_K^p is first-order definable if K is of characteristic p or if K contains a primitive p-th root ζ_p of unity and, if $p = 2$, the residue field Kv_K^p is not Euclidean. The gap occurred in the case where (K, v_K^p) is of mixed characteristic $(0, p)$ (i.e., $\mathrm{char}(K) = 0$ and $\mathrm{char} Kv_K^p = p$). However, we also present a slightly different proof to the (incomplete) one in [5].

To phrase the true definability result for v_K^p we should also take care of cases where v_K^p is, as it were, only definable 'by accident', that is, definable for the wrong reason. For example, there might be another prime $q \neq p$ with $v_K^q = v_K^p$, where v_K^q is 'truly' definable, but v_K^p is not. To pin this down we say that v_K^p is *$\emptyset$-definable as such* if there is a parameter-free $\mathcal{L}_{\mathrm{ring}}$-formula $\phi(x)$ such that, for all fields L elementarily equivalent to K in $\mathcal{L}_{\mathrm{ring}}$ (which we denote by $L \equiv K$), $\mathcal{O}_{v_L^p} = \phi(L)$. With this terminology we not only get a precise criterion for true (= 'as such') definability of v_K^p, but also the most uniform definition of v_K^p that one could wish for: a single $\mathcal{L}_{\mathrm{ring}}$-formula $\phi_p(x)$ does it for all of them:

Main Theorem. *For each prime p there is a parameter-free $\mathcal{L}_{\mathrm{ring}}$-formula $\phi_p(x)$ such that for any field K with either* $\mathrm{char}(K) = p$ *or $\zeta_p \in K$ the following are equivalent:*

(1) *ϕ_p defines v_K^p as such.*
(2) *v_K^p is $\emptyset$-definable as such.*
(3) *$p \neq 2$ or Kv_K^p is not Euclidean.*

The paper is organized as follows. We recall well-known definitions and facts about p-henselian valuations in the second section. In the third section, we give our Main Theorem and draw some conclusions from it. The Main Theorem is then proven in Section 4. Finally, we apply the Main Theorem to t-henselian fields in the last section. Improving a result of Koenigsmann (Theorem 4.1 in [4]), we show that any t-henselian field which is neither separably closed nor real closed admits a definable valuation inducing the (unique) t-henselian topology.