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The comparison of various club guessing principles

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ABSTRACT

We investigate variations of the club guessing principle, and show that most of the trivial implications cannot be reversed.

0. Introduction

R. Jensen introduced the first guessing principle $\diamondsuit$ in [4]. It asserts the sequence $\langle A_\delta : \delta < \omega_1 \rangle$ such that for every $\delta < \omega_1$, $A_\delta \subseteq \delta$ and for every subset X of ω_1, there exist a stationary set of $\delta < \omega_1$ such that $X \cap \delta = A_\delta$. Since then, there have been many principles proposed and applied.

The club guessing principle, introduced by S. Shelah in [6], is one of the most important guessing principles. It is defined as follows.

Definition 0.1. Let S be a stationary subset of $\omega_1 \cap \mathrm{Lim}$. We say that a sequence $\vec{C} = \langle C_\delta : \delta \in S \rangle$ is a *guessing sequence* on S if and only if for every $\delta \in S$, C_δ is an unbounded subset of δ. When $S = \omega_1 \cap \mathrm{Lim}$, we simply say a guessing sequence on ω_1. This applies to other definitions.

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When X and Y are sets of ordinals, we say that X is almost contained in Y and write $X \subseteq^* Y$ if and only if for some $\zeta < \sup X$, $X \setminus \zeta \subseteq Y$.

Definition 0.2. (See S. Shelah [6].) Let S be a stationary subset of $\omega_1 \cap \mathrm{Lim}$. We say that a guessing sequence $\vec{C} = \langle C_\delta : \delta \in S \rangle$ is a *fully club guessing sequence* on S if and only if for every club subset D of ω_1, there exists a $\delta \in S$ such that $C_\delta \subseteq D$.

If we replace $C_\delta \subseteq D$ by $C_\delta \subseteq^* D$, then we get the definition of a *tail club guessing sequence* on S.

That is, the sequence only needs to guess club subsets D of ω_1 by $C_\delta \subseteq D$. It is proved by the author in [3] that the existence of a fully club guessing sequence on S is equivalent to the one of a tail club guessing sequence on S. The club guessing principle $\mathrm{CG}(S)$ is the assertion that there exists a fully (tail) club guessing sequence on S.

A particularly important fact is that there always exists a club guessing sequence on every regular cardinal $\geq \aleph_2$. However, a club guessing sequence on ω_1 is also an interesting tool, demonstrated for example by F. Hernández-Hernández and the author in [2].

Since then, many club guessing principles are proposed. In this paper, we shall prove that there are no non-trivial implications among the ones listed in Section 1 except that there is one implication that is neither proved nor disproved. For the proofs, we shall show various preservation lemmas.

Particularly, in Section 5, we define the notion of d-dimensional generalized club guessing sequence, which can express various club guessing principles. Then, for any d-dimensional generalized club guessing sequence $\vec{I}$, we define $\vec{I}$-properness, which guarantees that the poset preserves the club guessing property of $\vec{I}$. Then, we show that the preservation theorem for $\vec{I}$-proper forcing holds if $\vec{I}$ satisfies the condition named countable generatedness. This provides a very general framework for this kind of situation. It was used in Sections 6 and 7.

In Section 8, we shall deal with the Interval Hitting Principle, for which the argument in the previous paragraph does not work. The proof is done by defining the ω-proper like condition that allows to slightly change the components of the tower. See the section for more precise explanations.

I am grateful for P. Nyikos, who asked the questions that lead to this research.

1. Variations of club guessing principles

In this section, we shall define some variations of club guessing principles that will be compared each other in this paper.

Definition 1.1. Let $\vec{C} = \langle C_\delta : \delta \in S \rangle$ be a guessing sequence on a stationary subset S of $\omega_1 \cap \mathrm{Lim}$. If $\mathrm{otp}(C_\delta) = \alpha$ for a club subset of $\delta \in S$ for some ordinal α, we say that $\vec{C}$ has *order type* α. If $\mathrm{otp}(C_\delta) < \delta$ for every $\delta \in S$, then we say that $\vec{C}$ is *short*.

When $\mathrm{otp}(C_\delta) = \omega$ for a club subset of $\delta \in S$, we say that $\vec{C}$ is a *ladder system* on S.

Definition 1.2. Let S be a stationary subset of $\omega_1 \cap \mathrm{Lim}$. The *interval hitting principle* on S, denoted by $\mathrm{IHP}(S)$, asserts that there exists a ladder system $\langle C_\delta : \delta \in S \rangle$ on S such that for every club subset D of ω_1, there exists a $\delta \in S$ such that for all but finitely many $n < \omega$, $D \cap [C_\delta(n), C_\delta(n+1)) \neq \emptyset$, where $C_\delta(n)$ denotes the $(n+1)$-st element of C_δ.

Definition 1.3. (See S. Todorcevic [8].) Let $k \leq \omega$ and S a stationary subset of $\omega_1 \cap \mathrm{Lim}$. $\mho_k(S)$ is defined to be the principle that asserts the existence of a sequence $\langle f_\delta : \delta \in S \rangle$ such that for each $\delta \in \omega_1 \cap \mathrm{Lim}$, f_δ is a continuous function from δ into k and for every club subset D of ω_1, there exists a $\delta \in \omega_1 \cap \mathrm{Lim}$ such that for every $\zeta < \delta$, $f_\delta''(D \cap [\zeta, \delta)) = k$. Such a sequence is called a $\mho_k$-sequence on S.

$\mho(S)$ means $\mho_\omega(S)$, and $\mho$ means $\mho(\omega_1 \cap \mathrm{Lim})$.