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# Automorphisms of models of set theory and extensions of NFU

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ABSTRACT

In this paper we exploit the structural properties of standard and non-standard models of set theory to produce models of set theory admitting automorphisms that are well-behaved along an initial segment of their ordinals. NFU is Ronald Jensen's modification of Quine's 'New Foundations' Set Theory that allows non-sets (urelements) into the domain of discourse. The axioms AxCount, $\text{AxCount}_{\leq}$ and $\text{AxCount}_{\geq}$ each extend NFU by placing restrictions on the cardinality of a finite set of singletons relative to the cardinality of its union. Using the results about automorphisms of models of subsystems of set theory we separate the consistency strengths of these three extensions of NFU. More specifically, we show that NFU + AxCount proves the consistency of $\text{NFU} + \text{AxCount}_{\leq}$, and $\text{NFU} + \text{AxCount}_{\leq}$ proves the consistency of $\text{NFU} + \text{AxCount}_{\geq}$.



## 1. Introduction

In [17] Ronald Jensen introduces a weakening of Quine's 'New Foundations' (NF), which he calls NFU, by allowing urelements (non-sets) into the domain of discourse. Despite the innocuous appearance of this weakening Jensen, in the same paper, shows that NFU is equiconsistent with a weak subsystem of ZFC and unlike NF is consistent with both the Axiom of Choice and the negation of the Axiom of Infinity. In the early nineties Randall Holmes and Robert Solovay embarked upon the project of determining the relative consistency strengths of natural extensions of NFU and the strengths of these extensions relative to subsystems and extensions of ZFC. Fruits of work in this direction can be seen in [16,23,24] and [6] which pinpoint the exact strength of a variety of natural extensions of NFU relative to subsystems and extensions of ZFC.

Throughout this paper we will use NFU to denote the theory described by Jensen in [17] supplemented with both the Axiom of Choice and the Axiom of Infinity. We will study three extensions of NFU that are obtained by adding the Axiom of Counting (AxCount), $\text{AxCount}_{\leq}$ and $\text{AxCount}_{\geq}$ respectively. The first of

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these axioms was proposed in [22] to facilitate induction in NF. Both $\mathrm{AxCount}_{\leq}$ and $\mathrm{AxCount}_{\geq}$ are natural weakenings of AxCount that were introduced by Thomas Forster in his Ph.D. thesis [8]. The combined work of Rolland Hinnion [14] and Jensen [17] shows that $\mathrm{NFU} + \mathrm{AxCount}_{\leq}$ proves the consistency of NFU. We improve this result by showing that $\mathrm{NFU} + \mathrm{AxCount}_{\leq}$ proves the consistency of $\mathrm{NFU} + \mathrm{AxCount}_{\geq}$. We also show that $\mathrm{NFU} + \mathrm{AxCount}$ proves the consistency of $\mathrm{NFU} + \mathrm{AxCount}_{\leq}$.

The techniques developed in [17] coupled with the observations in [4] establish a strong link between models of NFU and models of subsystems of ZFC that admit non-trivial automorphism. In light of this connection and motivated by questions related to the strength of the theory $\mathrm{NFU} + \mathrm{AxCount}_{\leq}$ Randall Holmes asked the following:

**Question 1.1.** Is there model $\mathcal{M} \models \mathrm{ZFC}$ the admits an automorphism $j : \mathcal{M} \longrightarrow \mathcal{M}$ such that

(i) $j(n) \geq n$ for all $\mathcal{M} \models n \in \omega$,
(ii) $j(\alpha) < \alpha$ for some $\mathcal{M} \models \alpha \in \omega_1$?

As shown in Theorem 7.5, a model equipped with such an automorphism would yield a model $\mathrm{NFU} + \mathrm{AxCount}_{\leq}$ in which the set of infinite cardinal numbers is countable. In Sections 3 and 4 of this paper we construct models of subsystems of ZFC equipped with automorphisms that are well-behaved along initial segment of their ordinals. In Section 3 we will show that models of set theory admitting automorphisms that move no points down along an initial segment of their ordinals can be built from standard models of set theory. This result allows us to show that every complete consistent extension of ZFC has a model which does not move any ordinal down. We then show in Section 4 that models of set theory admitting automorphism which move no natural number down but do move a recursive ordinal down can be built from non-standard $\omega$-models of set theory. This allows us to give a positive answer to Question 1.1 even when $\omega_1$ is replaced by $\omega_1^{\mathrm{CK}}$.

In Section 5 we describe the set theory NFU. We will survey work in [17] and [4] which shows that models of NFU can be built from models of subsystems of ZFC that admit non-trivial automorphism. We also survey Holmes's adaption to NFU of techniques developed in [14]. These techniques show that subsystems of ZFC can be interpreted in extensions of NFU. In Sections 6 and 7 we apply the model theoretic results proved in Sections 3 and 4 to shed light on the strengths of the theories $\mathrm{NFU} + \mathrm{AxCount}_{\leq}$ and $\mathrm{NFU} + \mathrm{AxCount}_{\geq}$. It is in these sections that we separate the consistency strengths of $\mathrm{NFU} + \mathrm{AxCount}$, $\mathrm{NFU} + \mathrm{AxCount}_{\leq}$ and $\mathrm{NFU} + \mathrm{AxCount}_{\geq}$.

The study of models admitting automorphisms has also recently yielded model theoretic characterisations of set theories and other foundational theories. In [6] Ali Enayat provides an elegant characterisation of a large cardinal extension of ZFC in terms of the existence of a model of a weak subsystem of ZFC that admits a well-behaved automorphism. Enayat's work [7] proves similar characterisations for a variety of subsystems of second order arithmetic. We have intentionally organised the background and results relating to automorphisms of models of subsystems of ZFC into Sections 2, 3 and 4 so readers who are only interested in these results can skip Sections 5, 6 and 7.

## 2. Background

Throughout this article we will use $\mathcal{L}$ to denote the language of set theory. If $A, B, C, \ldots$ are new relation, function or constant symbols then we will use $\mathcal{L}_{A,B,C,\ldots}$ to denote the language obtained by adding $A, B, C, \ldots$ to $\mathcal{L}$.

Given an extension of the language of set theory $\mathcal{L}'$, we will often have cause to consider the Lévy hierarchy of $\mathcal{L}'$ formulae which we will denote $\Delta_0(\mathcal{L}')$, $\Sigma_1(\mathcal{L}')$, $\Pi_1(\mathcal{L}')$, .... If $T$ is an $\mathcal{L}'$-theory then we say that an $\mathcal{L}'$-formula $\phi$ is $\Delta_n^T$ if and only if $\phi$ is provably equivalent in $T$ to both a $\Sigma_n(\mathcal{L}')$ and a $\Pi_n(\mathcal{L}')$