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Guessing more sets

Pierre Matet

Université de Caen, CNRS, Laboratoire de Mathématiques, BP 5186, 14032 Caen Cedex, France

A R T I C L E I N F O

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Let κ be a regular uncountable cardinal, and λ a cardinal greater than κ with cofinality less than κ. We consider a strengthening of the diamond principle $\diamondsuit_{\kappa,\lambda}$ that asserts that any subset of some fixed collection of λ^+ elements of $P_\kappa(\lambda)$ can be guessed on a stationary set. This new principle, denoted by $\diamondsuit_{\kappa,\lambda}[\lambda^+]$, implies that the nonstationary ideal on $P_\kappa(\lambda)$ is not $2^{(\lambda^+)}$-saturated. We establish that if λ is large enough and there are no inner models with fairly large cardinals, then $\diamondsuit_{\kappa,\lambda}[\lambda^+]$ holds. More precisely, it is shown that if $2^{(\kappa^{\aleph_0})} \leq \lambda^+$ and both Shelah's Strong Hypothesis SSH and the Almost Disjoint Sets principle ADS_λ hold, then $\diamondsuit_{\kappa,\lambda}[\lambda^+]$ holds. The paper also contains ZFC results. Suppose for example that $2^\kappa \leq \lambda^+$, there is a strong limit cardinal τ with $\mathrm{cf}(\lambda) < \tau \leq \kappa$, and either κ is a successor cardinal greater than ρ^{+3}, where ρ is the largest limit cardinal less than κ, or κ is a limit cardinal and $\sigma^\kappa < \lambda < (\sigma^\kappa)^{+\kappa}$ for some cardinal $\sigma \geq 2$. Then $\diamondsuit_{\kappa,\lambda}[\lambda^+]$ holds.

1. Introduction

We start by recalling some definitions.

Definition 1.1. For a set A and a cardinal ρ, let $P_\rho(A) = \{a \subseteq A : |a| < \rho\}$.

Definition 1.2. An *ideal* on a set X is a nonempty collection J of subsets of X such that

- $P(A) \subseteq J$ for any $A \in J$;
- $A \cup B \in J$ whenever $A, B \in J$.

Let J be an ideal on X. J is *proper* if $X \notin J$. We let $J^+ = \{A \subseteq X : A \notin J\}$, $J^* = \{A \subseteq X : X \setminus A \in J\}$, and $J \mid A = \{B \subseteq X : B \cap A \in J\}$ for every $A \in J^+$. Given a set Y and $f : X \to Y$, we let $f(J) = \{B \subseteq Y : f^{-1}(B) \in J\}$. For a cardinal ρ, J is *ρ-complete* if $\bigcup Q \in J$ for any $Q \in P_\rho(J)$. For a cardinal ρ and

E-mail address: pierre.matet@unicaen.fr.

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$Y \subseteq P(X)$, J is *Y-ρ-saturated* if there is no $Q \subseteq J^+$ with $|Q| = \rho$ such that $A \cap B \in Y$ for any two distinct members A, B of Q. J is *ρ-saturated* (respectively, *weakly ρ-saturated*) if it is J-ρ-saturated (respectively, $\{\emptyset\}$-ρ-saturated).

For the remainder of this introduction, we let κ denote a regular uncountable cardinal, and λ a cardinal greater than or equal to κ.

Definition 1.3. NS_κ denotes the nonstationary ideal on κ.

Definition 1.4. For an ideal J on κ, $\diamondsuit_\kappa[J]$ asserts the existence of $s_\alpha \subseteq \alpha$ for $\alpha < \kappa$ such that $\{\alpha : s_\alpha = A \cap \alpha\} \in J^+$ for every $A \subseteq \kappa$.

For $S \in NS_\kappa^+$, $\diamondsuit_\kappa(S)$ means that $\diamondsuit_\kappa[NS_\kappa|S]$ holds.

$\diamondsuit_\kappa(\kappa)$ is abbreviated as $\diamondsuit_\kappa$.

Definition 1.5. $I_{\kappa,\lambda}$ denotes the noncofinal ideal on $P_\kappa(\lambda)$, and $NS_{\kappa,\lambda}$ the nonstationary ideal on $P_\kappa(\lambda)$.

An ideal J on $P_\kappa(\lambda)$ is *fine* if $I_{\kappa,\lambda} \subseteq J$.

By an ideal on $P_\kappa(\lambda)$ we will always mean a fine, proper ideal on $P_\kappa(\lambda)$.

Let us also recall the definition of the game ideals $NG^\mu_{\kappa,\lambda}$ which will play an important role in this paper.

Definition 1.6. Let μ be a regular infinite cardinal less than κ. For $A \subseteq P_\kappa(\lambda)$, $G^\mu_{\kappa,\lambda}(A)$ denotes the following two players game consisting of μ moves. At step $\alpha < \mu$, player I selects $a_\alpha \in P_\kappa(\lambda)$, and II replies by playing $b_\alpha \in P_\kappa(\lambda)$. The players must follow the rule that for $\beta < \alpha < \mu$, $b_\beta \subseteq a_\alpha \subseteq b_\alpha$. II wins just in case $\bigcup_{\alpha<\mu} a_\alpha \in A$.

$NG^\mu_{\kappa,\lambda}$ denotes the collection of all $B \subseteq P_\kappa(\lambda)$ such that II has a winning strategy in $G^\mu_{\kappa,\lambda}(P_\kappa(\lambda) \setminus B)$.

$NG^\omega_{\kappa,\lambda}$ is abbreviated as $NG_{\kappa,\lambda}$.

1.1. Are diamonds really a girl's best friends?

Definition 1.7. For an ideal J on $P_\kappa(\lambda)$, $\diamondsuit_{\kappa,\lambda}[J]$ asserts the existence of $s_a \subseteq a$ for $a \in P_\kappa(\lambda)$ such that $\{a : s_a = A \cap a\} \in J^+$ for every $A \subseteq \lambda$.

$\diamondsuit_{\kappa,\lambda}[NS_{\kappa,\lambda}]$ is abbreviated as $\diamondsuit_{\kappa,\lambda}$.

This two-cardinal version of Jensen's guessing principle $\diamondsuit_\kappa$ was introduced by Jech in [12]. Ketonen observed that $\diamondsuit_{\kappa,\lambda}[J]$ can be reformulated as the assertion that there are 2^λ sets in J^+ that are almost disjoint in a strong sense:

Fact 1.8. *(See [15].) $\diamondsuit_{\kappa,\lambda}[J]$ holds if and only if we may find $X_A \in J^+$ for $A \subseteq \lambda$ such that for all $A, B \subseteq \lambda$, $X_A \cap X_B \subseteq \{a \in P_\kappa(\lambda) : A \cap a = B \cap a\}$.*

The following easily follows.

Fact 1.9. *(See [12].) If $\diamondsuit_{\kappa,\lambda}[J]$ holds, then J is neither weakly $\lambda^{<\kappa}$-saturated, nor $I_{\kappa,\lambda}$-2^λ-saturated.*

We will see that the converse sometimes holds:

Theorem A. *(See Proposition 5.8.) Suppose that $2^\lambda = \lambda^{<\kappa}$ and J is normal and not weakly $\lambda^{<\kappa}$-saturated. Then $\diamondsuit_{\kappa,\lambda}[J]$ holds.*