Annals of Pure and Applied Logic 166 (2015) 1019–1036

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Tight stationarity and tree-like scales

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ARTICLE INFO

Article history:
Received 10 July 2014
Accepted 2 February 2015
Available online 1 June 2015

MSC:
primary 03E35, 03E04
secondary 03E05, 03E55

Keywords:
pcf theory
Mutual stationarity
Tight structure
Tree-like scale

ABSTRACT

Let κ be a singular cardinal of countable cofinality, $\langle \kappa_n : n < \omega \rangle$ be a sequence of regular cardinals which is increasing and cofinal in κ. Using a scale, we define a mapping μ from $\prod_n \mathcal{P}(\kappa_n)$ to $\mathcal{P}(\kappa^+)$ which relates tight stationarity on κ to the usual notion of stationarity on κ^+. We produce a model where all subsets of κ^+ are in the range of μ for some κ a singular. Using a version of the diagonal supercompact Prikry forcing, we obtain such a model where κ is strong limit. Then we construct a sequence of stationary sets that is not tightly stationary in a strong way, namely, its image under μ is empty. All of these results start from a model with a continuous tree-like scale on κ.

1. Introduction

In their study of the non-saturation of the nonstationary ideal on $[\kappa]^\omega$ for κ a singular cardinal, Foreman and Magidor [8] introduced two concepts of stationarity for singular cardinals (even those of countable cofinality): *mutual stationarity* and *tight stationarity*. Each of these notions is a property of sequences $\vec{S} = \langle S_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ where $S_\xi \subseteq \kappa_\xi$ and $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ is a sequence of regular cardinals cofinal in κ. Tight stationarity is a more tractable strengthening of mutual stationarity that admits analogues of results for the classical notion of stationarity for regular cardinals, namely Fodor's lemma and Solovay's splitting theorem (whether those results hold for mutual stationarity is an open problem, see [7]).

This paper explores a method to transfer results from the theory of stationary subsets of κ^+ to that of tightly stationary sequences on κ. We introduce a function μ which takes a sequence $\vec{S} = \langle S_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ to a subset of κ^+. The key property of μ is that it preserves stationarity in the sense that $\vec{S}$ is tightly stationary if and only if $\mu(\vec{S})$ is stationary (this requires certain assumptions, see Lemma 2.5 for a precise statement). This function μ will be defined from a scale, and makes sense if there is a scale on $\prod \kappa_\xi$ modulo the ideal of bounded subsets of κ.

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[1] The author is deeply grateful to his advisor Itay Neeman for his guidance and many suggestions which improved this work.

http://dx.doi.org/10.1016/j.apal.2015.05.004

The existence of μ is by itself enough for some connections between stationarity at κ^+ and tight stationarity at κ. For example, it can be used to derive the version of Fodor's lemma previously obtained by Foreman–Magidor [8] for tight stationarity at κ from the usual Fodor's lemma at κ^+; see Proposition 2.7.

But for other applications, we want to have an inverse for μ, in the following strong sense: for each $A \subseteq \kappa^+$ we want to have a sequence $\vec{S}$ so that $\mu(\vec{S}) = A$ and $\mu(\vec{S'}) = \kappa^+ \setminus A$, where $\vec{S'}$ is the sequence $S'_\xi = \kappa_\xi \setminus S_\xi$. Call $A \subseteq \kappa^+$ *careful* if there exists such a sequence $\vec{S}$. The notion of carefulness can be thought of as a symmetrical strengthening of being in the range of μ—Boolean operations on careful sets commute with μ, although this is not generally true for sets which are just in the range of μ. Consequently, μ gives a particularly useful connection between careful subsets of κ^+ and sequences of the kind considered for tight stationarity.

If every subset of κ^+ is careful, then we can transfer Solovay's splitting theorem on κ^+ to the context of tight stationarity on κ. Under this assumption, we obtain a new splitting result for tightly stationary sets (Proposition 2.8). We remark that Proposition 2.8 differs from the splitting theorem obtained by Foreman and Magidor in [8].

Although there are many situations in which there exists a non-careful subset of κ^+, the main constructions in this paper show that it is actually consistent for every subset of κ^+ to be careful. In Section 3, we use forcing to construct a model where every subset of κ^+ is careful. The construction succeeds when κ has cofinality which is either countable or indestructibly supercompact. The μ function here is defined from a scale which is *tree-like*, a useful property studied by Pereira [11].

In Section 4, we start with a supercompact cardinal κ and modify the construction of Section 3 so that in the extension, κ is a strong limit singular cardinal of countable cofinality and every subset of κ^+ is careful. Additionally, collapses can be interleaved into the construction so that κ is the least cardinal fixed point (i.e., the least κ with $\kappa = \aleph_\kappa$). This uses ideas from the diagonal supercompact Prikry forcing of Gitik–Sharon [9].

In Section 5, we address the question of whether there is always a sequence of stationary sets that is not tightly stationary. We prove that if the scale used to define μ is tree-like, then there is a sequence $\vec{S}$ such that S_ξ is stationary for every $\xi < \mathrm{cf}(\kappa)$ and $\mu(\vec{S}) = \emptyset$ (in fact, $\mu(\vec{S'}) = \kappa^+$, where $S'_\xi = \kappa_\xi \setminus S_\xi$). This shows in particular that there is a sequence of stationary subsets which is not tightly stationary, under the seemingly mild assumption of a continuous tree-like scale at κ.

2. Preliminaries

First we will define the terminology used in the introduction. Let κ be a singular cardinal, and $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ a sequence of regular cardinals cofinal in κ. Take $\theta = (2^{2^\kappa})^+$ and let $\mathcal{A}$ be an algebra on $H(\theta)$, i.e., a structure on $H(\theta)$ with countably many functions in the language. If $M \prec \mathcal{A}$ is an elementary substructure, then define the *characteristic function of* M as $\chi_M : \xi \mapsto \sup(M \cap \kappa_\xi)$. We say M is *tight* if $M \cap \prod_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi$ is cofinal in $\prod(M \cap \kappa_\xi)$.

Suppose $S_\xi \subseteq \kappa_\xi$ for all $\xi < \mathrm{cf}(\kappa)$. The sequence $\vec{S} = \langle S_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ is *mutually stationary* if for any algebra $\mathcal{A}$ on $H(\theta)$ there is $M \prec \mathcal{A}$ such that $\{\xi : \chi_M(\xi) \notin S_\xi\}$ is bounded in $\mathrm{cf}(\kappa)$ (we say that χ_M *meets* $\vec{S}$). The sequence $\vec{S}$ is *tightly stationary* if for every $\mathcal{A}$ on $H(\theta)$, a tight structure $M \prec \mathcal{A}$ as in the previous definition can be chosen.

For our purposes, a *scale* is a sequence $\langle f_\alpha : \alpha < \kappa^+\rangle$ which is increasing and cofinal in $(\prod_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi, <^*)$, where $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ are regular cardinals cofinal in κ and $f <^* g$ if and only if $\{\xi : f(\xi) \geq g(\xi)\}$ is bounded in $\mathrm{cf}(\kappa)$. Scales were previously considered in the context of mutual and tight stationarity in [5] and [6]. A basic result of pcf theory due to Shelah [12] says that for singular κ, there is an increasing sequence of regular cardinals $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ which carries a scale. The scales in this paper will always be continuous, which means that for every $\beta < \kappa^+$ of cofinality $> \mathrm{cf}(\kappa)$, if there is an exact upper bound for $\langle f_\alpha : \alpha < \beta\rangle$ (i.e., a $<^*$-upper bound g such that $\langle f_\alpha : \alpha < \beta\rangle$ is cofinal in $\prod_\xi g(\xi)$) then f_β is such a bound.