Annals of Pure and Applied Logic 166 (2015) 1037–1048

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

An example of an automatic graph of intermediate growth

Alexei Miasnikov [a,1], Dmytro Savchuk [b,*,2]

[a] Department of Mathematical Sciences, Stevens Institute of Technology, Castle Point, Hoboken, NJ, 07030, United States
[b] Department of Mathematics and Statistics, University of South Florida, 4202 E Fowler Ave, Tampa, FL 33620-5700, United States

ARTICLE INFO

Article history:
Received 20 June 2014
Accepted 29 January 2015
Available online 3 June 2015

MSC:
68Q45
20F65
20F10
03D05

Keywords:
Automatic graph
Automatic structure
Cayley automatic group
Intermediate growth
Schreier graph

ABSTRACT

We give an example of a 4-regular infinite automatic graph of intermediate growth. It is constructed as a Schreier graph of a certain group generated by 3-state automaton. The question was motivated by an open problem on the existence of Cayley automatic groups of intermediate growth.

0. Introduction

Automatic groups were formally introduced by Thurston in 1986 motivated by earlier results of Cannon [6] on properties of Cayley graphs of hyperbolic groups. The latter results, in turn, were motivated by the pioneering work of Dehn on word problem in surface groups. All automatic groups have solvable in a quadratic time word problem and have at most quadratic Dehn function. If, in addition, a group is bi-automatic, then it has solvable conjugacy problem. For survey on the main results about the class of automatic groups we refer the reader to the multi-author book [8].

However, the class of automatic groups has its limitations. First of all, many of groups that play an important role in geometric group theory are not automatic. These include finitely generated nilpotent groups that are not virtually abelian, Baumslag–Solitar groups $BS(p, q)$ (unless $p = 0$, $q = 0$, or $p = \pm q$),

* Corresponding author. Tel. +1 813 974 4989.
E-mail addresses: amyasnik@stevens.edu (A. Miasnikov), savchuk@usf.edu (D. Savchuk).

1 Partially Supported by the Marsden Fund of The Royal Society of New Zealand.
2 Partially Supported by the New Researcher Grant from University of South Florida Internal Awards Program.

http://dx.doi.org/10.1016/j.apal.2015.05.005
0168-0072/

non-finitely presented groups, infinite torsion groups, $SL_n(\mathbb{Z})$. It would be desirable to extend the class of automatic groups to some wider class while preserving the computational routines of automatic groups. Further, some of the very basic questions about the class of automatic groups have still not been solved despite considerable efforts by the mathematical community. For example, it is not known whether each automatic group is bi-automatic.

In view of the above arguments it was quite natural to search for possible generalizations of the class of automatic groups. Several papers offered different approaches. Combable groups share with automatic groups the fellow traveler property, but have weaker constraints on the language used in the definition. Bridson in [3] discusses the relation between these two classes. The geometric generalization of the class of automatic groups, so-called, asynchronously indexed-combable groups, was defined and studied by Gilman and Bridson in [4]. It uses indexed languages and covers the fundamental groups of all compact 3-manifold satisfying the geometrization conjecture. Unfortunately, this class looses certain important algorithmic features of automatic groups. Recently Brittenham and Hermiller [5] defined another related class of stackable groups. They show, in particular, that every shortlex automatic group, including every word hyperbolic group, is regularly stackable, and that each stackable group is finitely presented. The exact relationship between these two classes is not yet fully understood.

The notion of a Cayley automatic group was introduced and studied in [12] as a natural generalization of the class of automatic groups. It has been observed that the Cayley graphs of automatic groups are automatic with respect to special encoding, in the sense of the theory of automatic structures developed, in particular, by Khoussainov and Nerode [13]. This theory can be traced back to works of Hodgson in the end of 1970's – beginning of 1980's [10]. For a survey on the results in this theory we refer the reader to a paper by Rubin [17]. A natural way to generalize the notion of automatic groups would be to remove the condition on the encoding on Cayley graphs. In other words, a group is called Cayley automatic, if its Cayley graph is automatic.

The class of Cayley automatic groups retains many algorithmic properties of the class of automatic groups, but is much wider. In particular, it includes many examples of nilpotent and solvable groups, which are not automatic in the standard sense. Some of Cayley automatic groups are not finitely presented. For example, the restricted wreath product of a nontrivial finite group G by $\mathbb{Z}$ is Cayley automatic. Further, it was recently shown by Miasnikov and Šunić in [14] that there exist Cayley automatic groups that are not Cayley biautomatic, thus resolving an analogue of a longstanding question of the theory of automatic groups. At the same time, main algorithmic tools of automatic groups still work. In particular, the word problem in each Cayley automatic group can be decided in a quadratic time.

Even further generalization of Cayley automatic groups, was recently introduced and studied by Elder and Taback in [7]. For each class of languages $\mathcal{C}$ they define $\mathcal{C}$-graph automatic groups in exactly the same way as Cayley automatic groups with the difference that the formal languages used in the definition must belong to class $\mathcal{C}$. In particular, if $\mathcal{C}$ is the class of regular languages, one simply obtains the class of Cayley automatic groups. One of the motivations to consider other classes of languages is the fact proved in [7] that polynomial time word problem algorithm is still preserved if one replaces the class of regular languages by the class of counter languages.

This paper was motivated by the following natural question regarding possible limitations of the class of Cayley automatic groups.

Question 1. *Is there a Cayley Automatic group of intermediate growth?*

Recall that the growth function of a finitely generated group G with respect to a generating set S is a function $\gamma_{G,S}\colon \mathbb{N} \to \mathbb{N}$ such that $\gamma_{G,S}(n)$ is equal to the number of elements of G that can be expressed as a product of at most n elements of $S \cup S^{-1}$. More generally, the growth function $\gamma_{\Gamma,x}(n)$ of a locally finite graph Γ with respect to the selected base point x is a function such that $\gamma_{\Gamma,x}(n)$ is the number of