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Universality, optimality, and randomness deficiency [☆]

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A Martin-Löf test $\mathcal{U}$ is *universal* if it captures all non-Martin-Löf random sequences, and it is *optimal* if for every ML-test $\mathcal{V}$ there is a $c \in \omega$ such that $\forall n(\mathcal{V}_{n+c} \subseteq \mathcal{U}_n)$. We study the computational differences between universal and optimal ML-tests as well as the effects that these differences have on both the notion of layerwise computability and the Weihrauch degree of LAY, the function that produces a bound for a given Martin-Löf random sequence's randomness deficiency. We prove several robustness and idempotence results concerning the Weihrauch degree of LAY, and we show that layerwise computability is more restrictive than Weihrauch reducibility to LAY. Along similar lines we also study the principle RD, a variant of LAY outputting the precise randomness deficiency of sequences instead of only an upper bound as LAY.

1. Introduction

Hoyrup and Rojas [15] fix a universal Martin-Löf test and define a function to be *layerwise computable* if it is computable on Martin-Löf random inputs when given what essentially amounts to a bound for the input's randomness deficiency as advice. The ML-test $\mathcal{U} = (\mathcal{U}_n)_{n\in\omega}$ that Hoyrup and Rojas use to

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define layerwise computability has the special property that for every ML-test $\mathcal{V} = (\mathcal{V}_n)_{n\in\omega}$ there is a $c \in \omega$ such that $\forall n(\mathcal{V}_{n+c} \subseteq \mathcal{U}_n)$. Miyabe [19] studies these special, so called *optimal*, tests. If $\mathcal{U}$ and $\mathcal{V}$ are two optimal ML-tests, then it is straightforward to see that the notion of layerwise computability is the same when defined via $\mathcal{U}$ as it is when defined via $\mathcal{V}$. However, the following example (essentially due to Miyabe, though with a slightly different proof) shows that there are universal ML-tests that are not optimal.

Example 1.1. Let $\mathcal{U}$ be any universal ML-test. Define a test $\mathcal{V}$ via $\mathcal{V}_n = \bigcap_{i\leq n}\mathcal{U}_n$ for all $n \in \omega$, thus making the test a descending chain. $\mathcal{V}$ is also a universal ML-test, so $\lambda(\mathcal{V}_n) \neq 0$ for all n. On the other hand $\lim_n \lambda(\mathcal{V}_n) = 0$. Therefore there are infinitely many n with $\lambda(\mathcal{V}_{n+1}) < \lambda(\mathcal{V}_n)$. Assume for the sake of argument that there are infinitely many n with $\lambda(\mathcal{V}_{2n+1}) < \lambda(\mathcal{V}_{2n})$, and let I be the set of these n. (The case in which there are infinitely many n with $\lambda(\mathcal{V}_{2n}) < \lambda(\mathcal{V}_{2n-1})$ is analogous.)

Define an ML-test $\mathcal{W}$ via $\mathcal{W}_n = \mathcal{V}_{2n+1}$ for all n. Clearly $\mathcal{W}$ meets the effectivity and measure conditions for being an ML-test. As $\bigcap_{n\in\omega}\mathcal{W}_n = \bigcap_{n\in\omega}\mathcal{V}_n = \bigcap_{n\in\omega}\mathcal{U}_n$, $\mathcal{W}$ is also a universal ML-test.

Fix any $c \in \omega$. For any $n \in I$ with $n \geq c$, we have that

$$\mathcal{U}_{n+c} \supseteq \mathcal{V}_{n+c} \supseteq \mathcal{V}_{2n} \supsetneq \mathcal{V}_{2n+1} = \mathcal{W}_n,$$

which implies that $\mathcal{U}_{n+c} \nsubseteq \mathcal{W}_n$.

That is, for every c there are infinitely many n with $\mathcal{U}_{n+c} \nsubseteq \mathcal{W}_n$. Thus $\mathcal{W}$ is a universal ML-test that is not optimal. □

Miyabe [19] obtains a compelling computational difference between optimal ML-tests and universal ML-tests: by a result of Merkle, Mihailović, and Slaman [18], there is a universal ML-test $\mathcal{U}$ and a left-c.e. real α such that $\forall n(\lambda(\mathcal{U}_n) = 2^{-n}\alpha)$. Miyabe proves that no optimal ML-test is of this form.

This article presents further differences between optimal ML-tests and universal ML-tests. If $\mathcal{U}$ is an optimal ML-test, then for every ML-test there trivially is a function f (in fact, a computable function f) such that $\forall n(\mathcal{V}_{f(n)} \subseteq \mathcal{U}_n)$. In Section 3, we show that if $\mathcal{U}$ is universal but not optimal, then such an f need not exist; and furthermore that there exist universal ML-tests $\mathcal{U}$ and $\mathcal{V}$ such that functions f as above do indeed exist, but such that all of these f are difficult to compute.

In Section 4, we ask if the notion of layerwise computability remains the same if we allow it to be defined using *any*, possibly non-optimal, universal ML-test. The answer is negative. It is possible to construct universal ML-tests that distort the randomness deficiencies assigned by a given ML-test quite chaotically. Likewise, we study the difference between the class of layerwise computable functions and the class of *exactly layerwise computable* functions, where we say that a function on MLR is exactly layerwise computable if it is uniformly computable given an ML-random sequence and its randomness deficiency (not merely an upper bound for its randomness deficiency). We show that both classes are different by identifying a function that is exactly layerwise computable but not layerwise computable.

Brattka, Gherardi and Hölzl [8] define and study the Weihrauch degree of LAY, a function representing the mathematical task of determining an upper bound for the randomness deficiency of a given MLR sequence. In particular, they investigate how LAY interacts with MLR—the principle that generates sequences that are ML-random relative to its input—and the principle $\mathrm{C}_{\mathbb{N}}$—the choice principle on natural numbers. We continue the study of LAY in Section 5, where we show that, unlike the notion of layerwise computability, the Weihrauch degree of LAY does not depend on the choice of the universal ML-test used to define it. Moreover, we show that, up to Weihrauch degree, the problem of exactly determining a ML-random sequence's randomness deficiency is equivalent to merely determining an upper bound for its randomness deficiency. We show that the Weihrauch degree of LAY enjoys several idempotence properties, and we investigate the complexity of sets that can be reduced to LAY.