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Interaction graphs: Additives

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ABSTRACT

Geometry of Interaction (GoI) is a research program initiated by Jean-Yves Girard which aims at defining a semantics of linear logic proofs accounting for the dynamical aspects of cut elimination. We present here a parametrised construction of a Geometry of Interaction for Multiplicative Additive Linear Logic (MALL) in which proofs are represented by families of directed weighted graphs. Contrarily to former constructions dealing with additive connectives [15,21], we are able to solve the known issue of obtaining a denotational semantics for MALL by introducing a notion of observational equivalence. Moreover, our setting has the advantage of being the first construction dealing with additives where proofs of MALL are interpreted by finite objects. The fact that we obtain a denotational model of MALL relies on a single geometric property, which we call the *trefoil property*, from which we obtain, for each value of the parameter, *adjunctions*. We then proceed to show how this setting is related to Girard's various constructions: particular choices of the parameter respectively give a combinatorial version of his latest GoI [21], a refined version of older Geometries of Interaction [13,12,15], and even a generalisation of his *multiplicatives* [11] construction. This shows the importance of the *trefoil property* underlying our constructions since all known GoI constructions to this day rely on particular cases of it.

1. Introduction

The Geometry of Interaction program [14] It was introduced by Girard a couple of years after his discovery of Linear Logic [10]. It aims at giving a semantics of linear logic proofs that would account for the dynamical aspects of cut-elimination, hence of computation through the proofs-as-program correspondence. Informally, a Geometry of Interaction (GoI) consists in:

- a set of mathematical objects — paraproofs — that will contain, among other things, the interpretations of proofs (or λ-terms);

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- a notion of execution that will represent the dynamics of cut-elimination (or β-reduction [6,27]).

Then, from these basic notions, one should be able to "reconstruct" the logic from the way the paraproofs interact:

- From the notion of execution, one defines a notion of orthogonality between the paraproofs that will allow to define formulas — types — as sets of paraproofs closed under bi-orthogonality (a usual construction in realisability). The notion of orthogonality should be thought of as a way of defining negation based on its computational effect.
- The connectives on formulas are defined from "low-level" operations on the paraproofs, following the idea that the rules governing the use of a connective should be defined by the way this connective acts at the level of proofs, i.e. by its computational effect.

Throughout the years, Girard defined several such semantics, mainly based on the interpretation of a proof as an operator on an infinite-dimensional Hilbert space. In particular, two such constructions offer a treatment of additive connectives of linear logic [15,21]. It is also worth noting that the first version of GoI [13] was used to analyse lambda-calculus' β-reduction [24,7], elucidating Lamping's optimal reduction [26].

The latest version of GoI [21], from which this work is greatly inspired, is related to quantum coherent spaces [19], which suggest future applications to quantum computing. Moreover, the great generality and flexibility of the definition of exponentials also seem promising when it comes to the study of complexity. Some results in this direction were already obtained: using a new technique proposed by Girard [23], the author obtained, in a joint work with Clément Aubert, new characterisations of the computational complexity classes **co-NL** [4] and **L** [5] as sets of operators in the hyperfinite type II_1 factor.

Interaction graphs Departing from the realm of infinite-dimensional vector spaces and linear maps between them, we propose a graph-theoretical GoI where proofs are interpreted by finite objects.[2] In this framework, it is possible to define the multiplicative and additive connectives of Linear Logic. Although not the first such work proposing a combinatorial formulation of GoI constructions [7,1,2], Interaction Graphs is the first work providing such an approach to Girard's hyperfinite GoI [21]. Another novelty lies in the fact that the construction is parametrised by a map from the interval $]0,1]$ to $\mathbb{R}_{\geqslant 0} \cup \{\infty\}$, and therefore yields not just one but a whole family of models.

We will show how, from any of these models, one can obtain a $*$-autonomous category with $⅋ \ncong \otimes$ and $1 \ncong \bot$, i.e. a non-degenerate denotational semantics for Multiplicative Linear Logic (MLL). However, as in all the versions of GoI dealing with additive connectives, our construction of additives does not define a categorical product. We solve this issue by introducing a notion of *observational equivalence* within the model. We are then able to define a categorical product from our additive connectives when considering classes of observationally equivalent objects, obtaining a denotational semantics for Multiplicative Additive Linear Logic (MALL).

One important point in this work is the fact that all results rely on a single geometric property we call the *trefoil property*. This property ensures the following four facts:

- we obtain a $*$-autonomous category; this is a consequence of the *three-term adjunction* obtained as a corollary of the trefoil property;
- the observational equivalence is a congruence on this category;
- the quotiented category inherits the $*$-autonomous structure;
- the quotiented category has a full subcategory with products.

[2] Even though the graphs we consider can have an infinite set of edges, linear logic proofs are represented by finite graphs (disjoint unions of transpositions).