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# Cone avoidance and randomness preservation

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ABSTRACT

Let $X$ be an infinite sequence of 0's and 1's. Let $f$ be a computable function. Recall that $X$ is strongly $f$-random if and only if the a priori Kolmogorov complexity of each finite initial segment $\tau$ of $X$ is bounded below by $f(\tau)$ minus a constant. We study the problem of finding a PA-complete Turing oracle which preserves the strong $f$-randomness of $X$ while avoiding a Turing cone. In the context of this problem, we prove that the cones which cannot always be avoided are precisely the K-trivial ones. We also prove: (1) If $f$ is convex and $X$ is strongly $f$-random and $Y$ is Martin-Löf random relative to $X$, then $X$ is strongly $f$-random relative to $Y$. (2) $X$ is complex relative to some oracle if and only if $X$ is random with respect to some continuous probability measure.



## 1. Introduction

We prove several results concerning Martin-Löf randomness and partial randomness. A theme of our results is *randomness preservation*, i.e., the phenomenon that if $X$ is (partially) random then $X$ is (partially) random relative to certain Turing oracles.

The purpose of this introductory section is to provide context and motivation for our results. In Section 1.1 we review basis theorems in general and the Randomness Preservation Basis Theorem in particular. In Section 1.2 we discuss the problem of combining two or more basis theorems into one basis theorem. We present a result to the effect that K-triviality is the only obstacle to combining the Randomness Preservation Basis Theorem with the Cone Avoidance Basis Theorem. In Section 1.3 we present some other results

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concerning partial randomness relative to a Turing oracle. Among the partial randomness notions which we consider are $\mu$-randomness, strong $f$-randomness, autocomplexity, and complexity.

### *1.1. Basis theorems*

**Remark 1.1.** A basis theorem is a theorem of the following form:

> Let $P$ be a nonempty, effectively closed set in Euclidean space.
> Then, at least one point of $P$ is "close to being computable."

Or, instead of assuming that $P$ is an effectively closed set in Euclidean space, it suffices to assume that $P$ is an effectively closed set in an effectively locally compact metric space. See Definition 4.1 below.

**Remark 1.2.** It is well known that basis theorems play an important role in the foundations of mathematics. The foundational idea underlying these applications is that, even though it is not always possible to find a *computable* point with a desired property, it is nevertheless often possible to find a point which is "close to being computable," in various senses.

**Remark 1.3.** Several well known basis theorems may be summarized as follows. Let $P$ be a nonempty, effectively closed set in Euclidean space. Then, for each of the following properties, there exists $Z \in P$ such that the property holds.

1. $Z$ is low, i.e., $Z' \leq_{\mathrm{T}} 0'$. This is the *Low Basis Theorem*, 1972 [16].
2. $Z$ is of recursively enumerable Turing degree. This is the *R.E. Basis Theorem*, 1972 [15].
3. $Z$ is hyperimmune-free, i.e., $(\forall f \leq_{\mathrm{T}} Z)\ (\exists g \leq_{\mathrm{T}} 0)\ \forall n\ (f(n) < g(n))$. This is the *Hyperimmune-Free Basis Theorem*, 1972 [16].
4. $X \not\leq_{\mathrm{T}} Z$, where $X \not\leq_{\mathrm{T}} 0$ is given. This is the *Cone Avoidance Basis Theorem*, 1960 [12].
5. $Y \in \mathrm{MLR}^Z$, where $Y \in \mathrm{MLR}$ is given. This is the *Randomness Preservation Basis Theorem*, 2005 [9,27].

Here $\leq_{\mathrm{T}}$ denotes Turing reducibility, $'$ denotes the Turing jump operator, $\mathrm{MLR} = \{Y \mid Y$ is Martin-Löf random$\}$, and $\mathrm{MLR}^Z = \{Y \mid Y$ is Martin-Löf random relative to $Z\}$.

**Remark 1.4.** The Cone Avoidance Basis Theorem is so named[3] because $Z$ avoids the Turing cone above $X$. This theorem has been applied in foundational studies touching on set existence [12], Turing degrees of complete theories [16,28,29], nonstandard models of arithmetic [17], Scott sets [1, Chapter XIX], and models of $\mathrm{WKL}_0$ (see [31, §§VIII.2, IX.2] and [33, §§9, 10]).

**Remark 1.5.** The Randomness Preservation Basis Theorem is so named because $Z$ preserves the randomness of $Y$. This theorem has been applied in the study of algorithmic randomness [3,27] and of formal systems for nonstandard measure theory [36]. There is also a less well known basis theorem concerning preservation of strong $f$-randomness [13, Theorem 4.6].

### *1.2. Combining basis theorems*

**Remark 1.6.** The question arises:

> Which basis theorems can be combined with each other?

Regarding this question, a large amount of information is available.

[3] The names "Cone Avoidance Basis Theorem" and "Randomness Preservation Basis Theorem" are our own terminology.