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Dimension spectra of random subfractals of self-similar fractals

Xiaoyang Gu [a], Jack H. Lutz [b,1,2], Elvira Mayordomo [c,*,3,4], Philippe Moser [d,3]

[a] *Twitter, Inc. 1355 Market St, Suite 900, San Francisco, CA 94103, USA*
[b] *Department of Computer Science, Iowa State University, Ames, IA 50011, USA*
[c] *Departamento de Informática e Ingeniería de Sistemas, Instituto de Investigación en Ingeniería de Aragón, Universidad de Zaragoza, 50018 Zaragoza, Spain*
[d] *Department of Computer Science, National University of Ireland, Maynooth, Maynooth, Co. Kildare, Ireland*

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A B S T R A C T

The dimension of a point x in Euclidean space (meaning the constructive Hausdorff dimension of the singleton set $\{x\}$) is the *algorithmic information density* of x. Roughly speaking, this is the least real number $\dim(x)$ such that $r \times \dim(x)$ bits suffice to specify x on a general-purpose computer with arbitrarily high precision 2^{-r}. The *dimension spectrum* of a set X in Euclidean space is the subset of $[0, n]$ consisting of the dimensions of all points in X.

The dimensions of points have been shown to be geometrically meaningful (Lutz 2003 [16], Hitchcock 2005 [12]), and the dimensions of points in self-similar fractals have been completely analyzed (Lutz and Mayordomo 2008 [18]). Here we begin the more challenging task of analyzing the dimensions of points in random fractals. We focus on fractals that are randomly selected subfractals of a given self-similar fractal. We formulate the specification of a point in such a subfractal as the outcome of an infinite two-player game between a *selector* that selects the subfractal and a *coder* that selects a point within the subfractal. Our selectors are algorithmically random with respect to various probability measures, so our selector–coder games are, from the coder's point of view, *games against nature*.

We determine the dimension spectra of a wide class of such randomly selected subfractals. We show that each such fractal has a dimension spectrum that is a closed interval whose endpoints can be computed or approximated from the parameters of the fractal. In general, the maximum of the spectrum is determined by the degree to which the coder can *reinforce* the randomness in the selector, while the minimum is determined by the degree to which the coder can *cancel* randomness in the selector. This constructive and destructive interference between the players' randomnesses is somewhat subtle, even in the simplest cases. Our proof techniques include van Lambalgen's theorem on independent random sequences, measure

* Corresponding author.
E-mail addresses: xiaoyanggu@gmail.com (X. Gu), lutz@cs.iastate.edu (J.H. Lutz), elvira@unizar.es (E. Mayordomo), pmoser@cs.nuim.ie (P. Moser).

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preserving transformations, an application of network flow theory, a Kolmogorov complexity lower bound argument, and a nonconstructive proof that this bound is tight.

1. Introduction

Fractals are inherently information-theoretic objects. The dimension n of a Euclidean space $\mathbb{R}^n$ is a measure of the amount of information (number of real numbers) that suffices to specify a point in $\mathbb{R}^n$ in a natural way. Similarly, the fact that the Hausdorff dimension of the Cantor "middle-thirds" set C is $\dim_H(C) = \log 2/\log 3 \approx 0.63$ tells us that it only takes about 0.63 of a real number to specify a point in C in a natural way. That is, roughly $(0.63)r$ bits suffice to specify the first r bits of a point in C. Intuitively, then, the Hausdorff (fractal) dimension $\dim_H(C)$ is an upper bound on the "information densities" of points in the fractal C.

Of course some points in the Cantor set can be specified even more concisely. The theory of constructive dimension, a computability-theoretic extension of Hausdorff dimension developed in the present century [16], assigns each *individual point* x in a Euclidean space $\mathbb{R}^n$ a *dimension* $\dim(x) \in [0, n]$ that is a measure of its information density. This notion of dimension has been shown to be geometrically meaningful. For example, if $X \subseteq \mathbb{R}^n$ is a "reasonably simple" set, in the sense that X is a union of Π^0_1 (i.e., computably closed) sets, then

$$\dim_H(X) = \sup_{x\in X} \dim(x), \tag{1.1}$$

which is a nonclassical, pointwise characterization of the classical Hausdorff dimensions of such sets [16,12].

The *self-similar fractals* form the best known and best understood class of fractals. (See Section 2.3 for a detailed review of self-similar fractals.) Each self-similar fractal F is given by an *iterated function system (IFS)* $S = (S_1, \ldots, S_{m-1})$ of *contracting similarities* S_i. A well-known theorem of Moran [20] states that

$$\dim_H(F) = \mathrm{sdim}(F) \tag{1.2}$$

holds for every self-similar fractal F, where $\mathrm{sdim}(F)$ is the *similarity dimension* of F. Much of the importance of this theorem arises from the fact that $\mathrm{sdim}(F)$ is easy to compute from the contraction ratios $c_0, \ldots, c_{m-1}$ of the respective similarities $S_1, \ldots, S_{m-1}$. That is, (1.2) gives an easy way to compute the Hausdorff dimensions of self-similar fractals.

The dimensions of points in computably self-similar fractals (those for which $S_1, \ldots, S_{m-1}$ are computable) have now been completely analyzed. If F is a self-similar fractal as above, then each point $x \in F$ is naturally given by at least one *coding sequence* $U \in \Sigma_m^\infty$, where $\Sigma_m = \{0, \ldots, m-1\}$. Intuitively, x is the result of a limiting process in which, at each stage $t \in \mathbb{N}$, we apply the contracting similarity $S_{U[t]}$. The main theorem of [18] says that, if F is computably self-similar, then, for each $x \in F$ and each coding sequence U for x,

$$\dim(x) = \mathrm{sdim}(F)\dim^{\pi_S}(U), \tag{1.3}$$

where $\dim^{\pi_S}(U) \in [0,1]$ is the dimension of the sequence $U \in \Sigma_m^\infty$ with respect to the *similarity probability measure* π_S on Σ_m, which arises from the IFS S in a natural manner. (This is a constructive version of Billingsley dimension [3] introduced in [18].)