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The equivalence of bar recursion and open recursion ☆

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A B S T R A C T

Several extensions of Gödel's system T with new forms of recursion have been designed for the purpose of giving a computational interpretation to classical analysis. One can organise many of these extensions into two groups: those based on *bar recursion*, which include Spector's original bar recursion, modified bar recursion and the more recent products of selections functions, or those based on *open recursion* which in particular include the symmetric Berardi–Bezem–Coquand (BBC) functional. We relate these two groups by showing that both open recursion and the BBC functional are primitive recursively equivalent to a variant of modified bar recursion. Our results, in combination with existing research, essentially complete the classification up to primitive recursive equivalence of those extensions of system T used to give a direct computational interpretation to choice principles.

1. Introduction

In a landmark paper of 1962, C. Spector extended Gödel's Dialectica interpretation of Peano arithmetic to countable choice and hence classical analysis by adding to the primitive recursive functionals a novel form of recursion now known as *bar recursion* [29]. In its broadest sense, bar recursion is a generalisation of primitive recursion to well founded trees, and can be informally described by the scheme

$$B^T(s) = \begin{cases} G(s) & \text{if } s \text{ is a leaf of } T \\ H(s, \lambda x \,.\, B^T(s * x)) & \text{otherwise.} \end{cases}$$

Spector's original bar recursion is just one of the first concrete instances of this kind of recursion, and many new variants have been developed in a number of different contexts. For example, in proof theory novel instances of bar recursion feature in [1] and [5] in order to give new realizability interpretations to countable dependent choice, while in computability theory a form of bar recursion was used in [18] to exhibit a continuous functional with a recursive associate which is nevertheless not S1–S9 computable. More recently,

☆ The results of this paper form Chapter 12 of the author's PhD dissertation [25].

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bar recursion in the form of products of selection functions has been shown to have deep connections with game-theory and the computation of so-called generalised Nash equilibria in unbounded sequential games [14,15]. The relationship between these many variants of bar recursion has been thoroughly investigated in [6,8,17,20,24].

In 1998, Berardi et al. [1] proposed a beautiful alternative to bar recursion – referred to here as the BBC-functional (or just BBC) – in order to give an efficient computational interpretation to countable choice. Their idea was a symmetric form of recursion, in many ways analogous to bar recursion but in which recursive calls are made over extensions of finite partial functions as opposed to extensions of finite sequences. Despite its apparent simplicity, the behaviour of the BBC-functional is seemingly much harder to understand than bar recursion. An early attempt by Berger [2] to justify BBC in a standard domain-theoretic type structure resorted to a complex non-constructive argument involving Zorn's lemma. A more satisfactory framework was subsequently developed by the same author in [3] where it is shown that BBC is a simple instance of a general schema of open recursion over the lexicographic ordering – a computational analogue of the principle of open induction which in turn forms the contrapositive to the well-known minimal bad sequence argument. Nevertheless, in contrast to bar recursion, relatively little is understood about open recursive functionals, and in particular their relationship to bar recursion remains unknown.

In this article, we prove that both open recursion and even the apparently weaker BBC functional are primitive recursively equivalent to a class of bar recursive functionals that includes the modified bar recursion of [5] and the more recent implicit product of selection functions of [17]. As an immediate consequence we obtain a new proof of the totality of BBC and open recursion, and also verify that neither of these is S1–S9 computable in the total continuous functionals. More importantly, in combination with previous results, we essentially complete the classification of the well-known computational interpretations of analysis according to primitive recursive equivalence. Moreover, we give direct constructions of open recursion and BBC as single instances of bar recursion, and vice versa, and in doing so hopefully shed some light on the qualitative relationship between these functionals, which may in turn lead to an improved understanding of how they compare in practice when used to extract realizers from proofs.

The organisation of this paper is fairly straightforward. In the remainder of this section we provide some essential preliminary material, before moving onto a brief survey of bar recursion and open recursion in Section 2. Sections 3 and 4 contain our main results: the definability of open recursion from bar recursion and the definability of bar recursion from the BBC-functional, respectively.

1.1. Heyting arithmetic in all finite types

In this paper we study recursion over all finite types. Our formulation of the finite types contains base types $\mathbb{N}$ and $\mathbb{B}$ for natural numbers and booleans, function types $\rho \to \tau$ (which we sometimes denote as τ^ρ), product types $\rho \times \tau$ and finite sequence types ρ^*. We will also make use of a type $\bar{\rho} \equiv \rho + 1$ in order to represent partial sequences as the type $\bar{\rho}^{\mathbb{N}}$ (we assume that it is decidable whether or not a given point is in the domain of a partial sequence). We write $x\colon\rho$ or x^ρ when x has type ρ.

A *discrete* type is any type that can be encoded in $\mathbb{N}$. For example, all of $\mathbb{B}$, $\mathbb{N}$ and $\mathbb{B} \times \mathbb{N}$ are discrete, but $\mathbb{N} \to \mathbb{N}$ is not. The topological significance of the discrete types in the context of the continuous functionals is discussed in [11]. The restriction on some types being discrete will be important later in order to ensure that the defining equations of certain recursors are consistent with Heyting arithmetic.

We work in the standard theory $\mathsf{E\text{-}HA}^\omega$ of fully extensional Heyting arithmetic in all finite types (see e.g. [21,30] for details), which contains variables and quantifiers for all types, induction for arbitrary formulas, and the usual non-logical constants with their defining axioms, including symbols R_ρ for primitive recursion in all finite-types: