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The bounded proof property via step algebras and step frames

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ABSTRACT

The paper introduces semantic and algorithmic methods for establishing a variant of the analytic subformula property (called 'the bounded proof property', bpp) for modal propositional logics. The bpp is much weaker property than full cut-elimination, but it is nevertheless sufficient for establishing decidability results. Our methodology originated from tools and techniques developed on one side within the algebraic/coalgebraic literature dealing with free algebra constructions and on the other side from classical correspondence theory in modal logic. As such, our approach is orthogonal to recent literature based on proof-theoretic methods and, in a way, complements it.

We applied our method to simple logics such as **K**, **T**, **K4**, **S4**, etc., where establishing basic metatheoretical properties becomes a completely automatic task (the related proof obligations can be instantaneously discharged by current first-order provers). For more complicated logics, some ingenuity is still needed, however we were able to successfully apply our uniform method to the well-known cut-free system for **GL**, to Goré's cut-free system for **S4.3**, and to Ohnishi–Matsumoto's analytic system for **S5**.

1. Introduction

We revisit the method of describing free algebras of modal logics by approximating them with finite partial algebras. This construction is longstanding, but here we apply it for investigating proof-theoretic aspects of modal logics. The key points of the method are that every free algebra is approximated by a chain of partial algebras: the n-th algebra in the chain interprets the formulae of modal complexity n, i.e., the formulae having at most n nested modal operators. Since these algebras are finite, the dual spaces of these approximants are just sets that can be described explicitly [1],[3] [31]. In a sense, the basic idea of this construction can be traced back to [29]. In [30] this method was applied to free Heyting algebras. In recent

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[3] The original talk was given at the BCTCS in 1988.

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years there has been a renewed interest towards this construction of free algebras via finite approximations e.g., [12,15,16,28,32].

In this paper[4] we study proof-theoretic consequences of this method for axiomatic systems of modal logic. In particular, we will concentrate on the *bounded proof property*. An axiomatic system Ax has the bounded proof property (bpp, for short) if every formula ϕ of modal complexity at most n derived in Ax from some set Γ containing only formulae of modal complexity at most n, can be derived from Γ in Ax only using formulae of modal complexity at most n. The bounded proof property is a kind of analytic subformula property limiting the proof search space. It is an indicator of the robustness of a proof system, and hence is desirable to have. This property holds for proof systems enjoying the subformula property (the latter usually follows from sequent calculi admitting cut elimination). The bounded proof property depends on an axiomatization of a logical system. That is, one axiomatization of a logic may have the bpp and the other not (examples and counterexamples will be given in Section 8 of this paper).

The main tools we employ in the paper are the one-step frames introduced in [28] and [15]. A one-step frame is a two-sorted structure which admits interpretations of modal formulae without nested modal operators. Thus, one-step frames supply a suitable semantics and suitable duals of the approximating algebras of formulae of bounded complexity. We show that an axiomatic system Ax for a logic L has the bpp and the finite model property (fmp, for short) iff every one step-frame validating Ax is a p-morphic image of a finite Kripke frame for L. This gives a purely semantic characterization of the bpp. The main advantage of this criterion is that it is relatively easy to verify. In Section 1.1 below, we give an example explaining the details of our machinery step-by-step. Here we summarize the main ingredients. Given an axiom of a modal logic, we apply to it the following procedure:

(I) We rewrite it into a one-step rule, that is, a rule of modal complexity 1. One-step rules can be interpreted in one-step frames.

(II) Second, we use an analogue of the classical correspondence theory, to obtain a first-order condition (or a condition of first-order logic enriched with fixed-point operators) for a one-step frame corresponding to the one-step rule.

(III) Finally, we need to find a standard frame p-morphically mapped onto any finite one-step frame satisfying this first-order condition. This part is not automatic, but we have some standard templates to define a procedure modifying the relation of a one-step frame so that the obtained frame is standard (Kripke).

In easy cases, e.g., for modal logics such as **K**, **T**, **K4**, **S4**, the frame we obtain by the templates in (III) is a frame of the logic and is p-morphically mapped onto the original one-step frame. The bpp and fmp for these logics follow by our criterion. For more complicated systems such as **S4.3**, **S5** and **GL**, we show that the rules that we automatically obtain from some standard axiomatizations are not good—we prove that these axiomatizations do not have the bpp. However, we also show using our method, that cut-free rules for **GL** [54–56,2], Goré's cut-free rules for **S4.3** [37], and Ohnishi–Matsumoto's analytic rules for **S5** [47] provide axiomatic systems having the bpp.

In order to explain the basic idea of our technique, we proceed by detailing a rather simple (but still significant) example.

1.1. A worked out example

Consider the modal logic obtained by adding to the basic normal modal system **K** the 'density' axiom:

[4] This paper is an extended version of [13].