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First order $S4$ and its measure-theoretic semantics

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ABSTRACT

The first order modal logic *FOS*4 is a combination of the axioms and rules of inference of propositional $S4$ and classical first order logic with identity. We give a topological and measure-theoretic semantics for *FOS*4 with expanding domains. The latter extends the measure-theoretic semantics for propositional $S4$ introduced by Scott and studied in [3,6], and [8]. The main result of the paper is that *FOS*4 is complete for the measure-theoretic semantics with countable expanding domains. More formally, *FOS*4 is complete for the Lebesgue measure algebra, $\mathcal{M}$, or algebra of Borel subsets of the real line modulo sets of measure zero, with countable expanding domains. A corollary to the main result is that first order intuitionistic logic *FOH* is complete for the frame of open elements in $\mathcal{M}$ with countable expanding domains. We also show that *FOS*4 is *not* complete for the real line or the infinite binary tree with limits with countable expanding domains.

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1. Introduction

It is well-known that the propositional modal logic $S4$ can be interpreted in topological spaces. In the topological semantics, formulas are assigned to subsets of a fixed topological space, and the $\Box$-modality is interpreted by the topological interior. As early as 1944, McKinsey and Tarski [9] showed that $S4$ is complete for the real line, the rationals, Cantor space, and many 'nice' metric topologies.[1] In recent years, Scott showed that in addition to the topological semantics, we can interpret $S4$ in the *Lebesgue measure algebra*, or algebra of Borel subsets of the real line modulo sets of measure zero. Here each propositional variable is assigned to an element of the algebra, instead of to a subset of a topological space. Conjunctions, disjunctions and negations are interpreted by Boolean meets, joins and complements respectively, and we can construct an interior operator on the algebra that interprets the $\Box$-modality. The measure-theoretic

[1] These are special cases of McKinsey and Tarski's theorem in [9] that $S4$ is complete for any dense-in-itself separable metrizable space. Rasiowa and Sikorski improve the result in [10] showing that $S4$ is complete for any dense-in-itself metrizable space.

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semantics (as I'll refer to it) is in some ways reminiscent of the older topological semantics, particularly of topological interpretations over the real line. It was shown in [6] and [3] that $S4$ is complete for the Lebesgue measure algebra.[2]

Thus when it comes to propositional $S4$ we have many nice completeness theorems in both the topological and measure-theoretic semantics. But once we pass to *quantified* or *first order* modal logics, the landscape changes quite dramatically. In the topological semantics for quantified $S4$, we can think of each point in a topological space as carrying a first order model. Formulas are true or false at points in the topological space, and true for the model as a whole if true throughout the space. The topological semantics for quantified modal logics has been studied far less than its propositional counterpart, but what little we do know shows that completeness results for particular spaces are harder to come by. Take $QS4$, the quantified modal logic that combines the axioms and rules of inference of propositional $S4$ with the identity-free fragment of classical first order logic. It was shown by Rasiowa and Sikorski [10] that $QS4$ is *not* complete for any Baire space with countable constant domains. This result was recently extended by Kremer [5], who shows that $QS4$ is not complete for any locally connected space with arbitrary constant domains. In particular, $QS4$ is not complete for the real line with arbitrary constant domains. Perhaps the only 'nice' completeness result for a particular topological space is Kremer's proof in [5] that $QS4$ is complete for the rationals with countable constant domains.

Now Rasiowa and Sikorski [10], and Kremer [5] all study *constant* domain semantics for identity-free quantified modal languages. The focus of the present paper is on first order modal languages that contain identity, and the logic $FOS4$, which combines the axioms and rules of inference of propositional $S4$ with those of classical first order logic. In order to give a semantics for $FOS4$, we need to consider models in which the domain of individuals varies across points in space. Indeed, in the more familiar Kripke (or frame) semantics for quantified modal logics it is well known that there are many choices to be made as to whether and how domains can vary across possible worlds. In an 'expanding domain' semantics, the domain at a world is a subset of the domain of related worlds (equivalently, individuals can *come into* existence as one moves along the accessibility relation, but cannot go *out of* existence—hence 'expanding').[3] Are there analogs for this in the topological or measure-theoretic semantics? At first sight, it is not clear how there could be. The very idea of expanding domains is defined in terms of the accessibility relation in Kripke frames, a relation that has no place in the topological, much less algebraic semantics. But in fact there are natural analogs in both cases. Just as the accessibility relation interprets modality in Kripke semantics, so the topological interior or open sets interpret modality in the topological semantics. In the expanding domain Kripke semantics, we require that an object which exists at a world w exists at all worlds downstream from w; analogously, in the expanding domain topological semantics we require that an object that exists at a point x exists throughout an open neighborhood of x. (A similar constraint can be introduced in the algebraic—hence also measure-theoretic—semantics but we save the details for Section 8.) As we'll see below, the expanding domain topological semantics for $FOS4$ generalizes the expanding domain Kripke semantics for $FOS4$ in much the same way that the topological semantics for propositional $S4$ generalizes Kripke semantics for propositional $S4$.[4]

[2] The measure-theoretic semantics is an important special case of the algebraic semantics, where the algebra used to interpret the modal language carries a countably additive measure.

[3] Constraints on how individuals may vary across possible worlds are tied to the verification of the Barcan and converse Barcan formulas. The expanding domain frame semantics verifies the converse Barcan formula, but not the Barcan formula. The constant domain semantics verifies both. See Section 3. It is interesting to note that the constant domain topological semantics does *not* verify the Barcan formula, hence introducing varying domains in the topological semantics is not necessary for the purpose of refuting that formula (it *is* necessary, however, for giving an adequate semantics for $FOS4$).

[4] It should be mentioned that there is an alternative 'topological sheaf' semantics for first order modal logics introduced by Awodey and Kishida in [1], which also extends the topological semantics for propositional $S4$. In the sheaf semantics, topological sheaves are constructed over a topological space in order to supply these spaces with variable domains. (A topological sheaf over X is a topological space F together with a local homeomorphism $\pi : F \to X$. Here F is a space of individuals, and an individual $d \in F$ 'lives' at the point $\pi(d) \in X$.) In the sheaf semantics, individuals that exist at a world do not exist at any other world, but may have 'counterparts' in other worlds. Although the machinery of the sheaf semantics is quite different from that of the expanding

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