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On effectively closed sets of effective strong measure zero

Kojiro Higuchi [a,*], Takayuki Kihara [b]

[a] *Department of Mathematics and Informatics Faculty of Science, Chiba University, 1-33 Yayoi-cho, Inage, Chiba, 263-8522, Japan*
[b] *Japan Advanced Institute of Science and Technology, Nomi, Ishikawa 923-1292, Japan*

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ABSTRACT

The strong measure zero sets of reals have been widely studied in the context of set theory of the real line. The notion of strong measure zero is straightforwardly effectivized. A set of reals is said to be of *effective strong measure zero* if for any computable sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive rationals, a sequence of intervals I_n of diameter ε_n covers the set. We observe that a set is of effective strong measure zero if and only if it is of measure zero with respect to any outer measure constructed by Monroe's Method from a computable atomless outer premeasure defined on all open balls. This measure-theoretic restatement permits many characterizations of strong measure zero in terms of semimeasures as well as martingales. We show that for closed subsets of Cantor space, effective strong nullness is equivalent to another well-studied notion called *diminutiveness*, the property of not having a computably perfect subset. Further, we prove that if P is a nonempty effective strong measure zero Π^0_1 set consisting only of noncomputable elements, then some Martin-Löf random reals compute no element in P, and P has an element that computes no autocomplex real. Finally, we construct two different special Π^0_1 sets, one of which is not of effective strong measure zero, but consists only of infinitely-often K-trivial reals, and the other is perfect and of effective strong measure zero, but contains no anti-complex reals.

1. Introduction

1.1. Background

Miniaturization of set-theoretic notions is sometimes useful in computability theory. For example, set-theoretic forcing is transformed into a notion called arithmetical forcing and n-generic reals, which has become a fundamental tool in computability theory. There is another set-theoretical notion whose miniaturization we expect to play an important role. The notion is known as *strong measure zero* which was

* Corresponding author. Tel.: +81 43 251 1111.
E-mail addresses: khiguchi@g.math.s.chiba-u.ac.jp (K. Higuchi), kihara.takayuki.logic@gmail.com (T. Kihara).

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introduced by Émile Borel in 1919. Careful consideration of the measure theoretic behavior of sets of reals has profound significance in the study of algorithmic randomness [14,30]. Binns [5–7] conducted a deep study of notions stronger than being of measure zero/Hausdorff dimension zero, and clarified an interesting connection among such measure theoretic smallness, Muchnik degrees, and Kolmogorov complexity.

In his thesis in 2011, Kihara pointed out the relationship between Binns' smallness properties [5–7] and the notion of small sets in set theory of the real line [9]. Kihara introduced the notion of *effective strong measure zero* to formalize his idea. In Section 2, we will see that a set of reals is of effective strong measure zero if and only if for any computable atomless[1] outer measure defined on all open balls, the set is of measure zero with respect to the outer measure. This characterization urges us to study other effectivizations of strong measure zero. As one such effectivization, we study *strong Martin-Löf measure zero* introduced in a personal communication between Kihara and Miyabe in 2012. A set of reals is called strong Martin-Löf measure zero if for any computable atomless outer premeasure defined on all open balls, the set contains no Martin-Löf random real with respect to the outer measure induced by the premeasure.

It is known that the notion of Martin-Löf randomness (nullness, and Martin-Löf nullness) admits many natural characterizations such as incompressibility (in terms of Kolmogorov complexity) and unpredictability (in terms of martingales). In Section 2, we will focus on characterizations of Martin-Löf randomness by semimeasures, Kolmogorov complexity, and martingales, and extend such characterizations to Martin-Löf nullness with respect to any outer measure induced by a computable outer premeasure. This leads to the conclusion that the concept of effective strong measure zero is robust enough to have many characterizations just as in the case of Martin-Löf reals.

In Section 3, we review the results of Higuchi/Kihara [21] in their research on the Π^0_1 sets of reals of effective strong measure zero as well as their Muchnik degrees. In contrast to Laver's model [27] of ZFC in which all strongly measure zero sets are countable, one can easily construct an effectively strongly measure zero set of reals that is uncountable and Π^0_1 definable. Indeed, the class of uncountable Π^0_1 definable effective strong measure zero subsets of Cantor space has nontrivial properties. We see that for closed sets of reals, effective strong measure zero is equivalent to another well-studied notion called *diminutiveness* [7], the property of not having a computably perfect subset. Further, we prove that if P is a nonempty effective strong measure zero Π^0_1 set consisting only of noncomputable elements, then some Martin-Löf random real computes no element in P, and P has an element that computes no autocomplex real. Here, an infinite binary sequence x is (*auto-*)*complex* if there exists an (x-)computable function f such that $\mathrm{K}(x \upharpoonright f(n)) \geq n$ for all $n \in \mathbb{N}$, where K denotes the prefix-free Kolmogorov complexity.

In Section 4, we see some interactions between measure theoretic smallness and Kolmogorov complexity. We prove two non-basis theorems for small Π^0_1 sets and very small Π^0_1 sets. By using the non-basis results, we construct a computably perfect Π^0_1 set consisting only of non-generic reals that are both complex and infinitely often K-trivial, and we also construct a perfect (but effectively strongly measure zero) Π^0_1 set consisting only of non-generic reals that are neither complex nor anti-complex. Here, an infinite binary sequence $x \in 2^{\mathbb{N}}$ is *infinitely often* K-*trivial* if there exists a constant c such that $\mathrm{K}(x \upharpoonright n) \leq \mathrm{K}(n) + c$ for infinitely many $n \in \mathbb{N}$, and x is *anti-complex* if there exists an (x-)computable function f such that $\mathrm{K}(x \upharpoonright f(n)) \leq n$ for all $n \in \mathbb{N}$.

1.2. Notation

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ denote the set of all natural numbers; $\mathbb{N}^{\mathbb{N}} = \{f \mid f : \mathbb{N} \to \mathbb{N}\}$, Baire space; $2^{\mathbb{N}} = \{f \mid f : \mathbb{N} \to \{0, 1\}\}$, Cantor space; $\mathbb{N}^{<\mathbb{N}}$, the set of all finite strings of natural numbers; and $2^{<\mathbb{N}}$, the

[1] A point which has a positive μ-measure is called an *atom* of μ. A measure having an atom is called *atomic*. Otherwise, it is called *atomless*. As pointed out by L.A. Levin in 1970, every computable real can be μ-random for a computable *atomic* probability measure μ. We avoid such a singular case by restricting the range of μ to atomless measures.