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How much randomness is needed for statistics?

Bjørn Kjos-Hanssen [a,*], Antoine Taveneaux [b], Neil Thapen [c]

[a] *University of Hawai'i at Mānoa, Honolulu, HI 96822, USA*
[b] *LIAFA, Université Paris Diderot – Paris 7, 75205 Paris Cedex 13, France*
[c] *Academy of Sciences of the Czech Republic, 115 67 Praha 1, Czech Republic*

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ABSTRACT

In algorithmic randomness, when one wants to define a randomness notion with respect to some non-computable measure λ, a choice needs to be made. One approach is to allow randomness tests to access the measure λ as an oracle (which we call the "classical approach"). The other approach is the opposite one, where the randomness tests are completely effective and do not have access to the information contained in λ (we call this approach "Hippocratic"). While the Hippocratic approach is in general much more restrictive, there are cases where the two coincide. The first author showed in 2010 that in the particular case where the notion of randomness considered is Martin-Löf randomness and the measure λ is a Bernoulli measure, classical randomness and Hippocratic randomness coincide. In this paper, we prove that this result no longer holds for other notions of randomness, namely computable randomness and stochasticity.

1. Introduction

In algorithmic randomness theory we are interested in which almost sure properties of an infinite sequence of bits are effective or computable in some sense. Martin-Löf defined randomness with respect to the uniform fair-coin measure μ on 2^ω as follows.

A sequence $X \in 2^\omega$ is *Martin-Löf random* if we have $X \notin \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$ for every sequence of uniformly Σ^0_1 (or effectively open) subsets of 2^ω such that $\mu(\mathcal{U}_n) \le 2^{-n}$.

Now if we wish to consider Martin-Löf randomness for a Bernoulli measure μ_p (that is, a measure such that the ith bit is the result of a Bernoulli trial with parameter $p \in [0,1]$), we have two possible ways to extend the previous definition.

* Corresponding author.
E-mail addresses: bjoernkh@hawaii.edu (B. Kjos-Hanssen), taveneaux@calculabilite.fr (A. Taveneaux), thapen@math.cas.cz (N. Thapen).

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The first option is to consider p as an oracle (with an oracle p we can compute μ_p) and relativize everything to this oracle. Then X is μ_p-Martin-Löf random if for every sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of uniformly $\Sigma_1^0[p]$ sets such that $\mu_p(\mathcal{U}_n) \le 2^{-n}$ we have $X \notin \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$. We will call this approach the *classical*[1] notion of Martin-Löf randomness relative to μ_p.

Another option is to keep the measure μ_p "hidden" from the process which describes the sequence $(\mathcal{U}_n)$. One can merely replace μ by μ_p in Martin-Löf's definition but still require $(\mathcal{U}_n)$ to be uniformly Σ_1^0 in the unrelativized sense. This notion of randomness was introduced by Kjos-Hanssen [7] who called it *Hippocratic randomness*; Bienvenu, Doty and Stephan [2] used the term *blind randomness*.

Kjos-Hanssen showed that for Bernoulli measures, Hippocratic and classical randomness coincide in the case of Martin-Löf randomness. Bienvenu, Gács, Hoyrup, Rojas and Shen [3] extended Kjos-Hanssen's result to other classes of measures. Here we go in a different direction and consider weaker randomness notions, such as computable randomness and stochasticity. We discover the contours of a dividing line for the type of betting strategy that is needed in order to render the probability distribution superfluous as a computational resource.

We view *statistics* as the discipline concerned with determining the underlying probability distribution μ_p by looking at the bits of a random sequence. In the case of Martin-Löf randomness it is possible to determine p ([7]), and therefore Hippocratic randomness and classical randomness coincide. In this sense, Martin-Löf randomness is sufficient for statistics to be possible, and it is natural to ask whether smaller amounts of randomness, such as computable randomness, are also sufficient.

Notation. Our notation generally follows Nies' monograph [13]. We write 2^n for $\{0,1\}^n$, and for sequences $\sigma \in 2^{\le\omega}$ we will also use σ to denote the real with binary expansion $0.\sigma$, that is, the real $\sum_{i=1}^{\infty} \sigma(i)2^{-i}$. We use ε to denote the empty word, $\sigma(n)$ for the nth element of a sequence and $\sigma \upharpoonright n$ for the sequence formed by the first n elements. For sequences ρ, σ we write $\sigma \prec \rho$ if σ is a proper prefix of ρ and denote the concatenation of σ and ρ by $\sigma.\rho$ or simply $\sigma\rho$. Throughout the paper we set $n' = n(n-1)/2$.

1.1. Hippocratic martingales

Formally a martingale is a function $\mathcal{M} : 2^{<\omega} \to \mathbb{R}^{\ge 0}$ satisfying

$$\mathcal{M}(\sigma) = \frac{\mathcal{M}(\sigma 0) + \mathcal{M}(\sigma 1)}{2}.$$

Intuitively, such a function arises from a betting strategy for a fair game played with an unbiased coin (a sequence of Bernoulli trials with parameter 1/2). In each round of the game we can choose our stake, that is, how much of our capital we will bet, and whether we bet on heads (1) or tails (0). A coin is tossed, and if we bet correctly we win back twice our stake.

Suppose that our betting strategy is given by some fixed function S of the history σ of the game up to that point. Then it is easy to see that the function $\mathcal{M}(\sigma)$ giving our capital after a play σ satisfies the above equation. On the other hand, from any $\mathcal{M}$ satisfying the equation we can recover a corresponding strategy S.

More generally, consider a biased coin which comes up heads with probability $p \in (0,1)$. In a fair game played with this coin, we would expect to win back $1/p$ times our stake if we bet correctly on heads, and

[1] The classical approach has actually two approaches. Reimann and Slaman [14, arXiv:0802.2705, Definition 3.2.] defined a real x to be μ-random if, for some oracle z computing μ, the real x is μ-random relative to z. Levin [9] and Gács [6] use a uniform test, which is a left-c.e. function $u : 2^\omega \times M(2^\omega) \to [0,\infty]$ such that $\int u(x,\mu)d\mu \le 1$ for all μ where $M(2^\omega)$ is the space of probability measures on 2^ω. Since there is a universal uniform test u_0, define x to be μ-random if $u_0(x,\mu) < \infty$. Day and Miller [4] showed that these approaches actually coincide.