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An application of proof mining to nonlinear iterations

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ABSTRACT

In this paper we apply methods of proof mining to obtain a highly uniform effective rate of asymptotic regularity for the Ishikawa iteration associated with nonexpansive self-mappings of convex subsets of a class of uniformly convex geodesic spaces. Moreover, we show that these results are guaranteed by a combination of logical metatheorems for classical and semi-intuitionistic systems.

1. Introduction

Proof mining is a paradigm of research concerned with the extraction of hidden finitary and combinatorial content from proofs that make use of highly infinitary principles. This new information is obtained after a logical analysis of the original mathematical proof, using proof-theoretic techniques called *proof interpretations*. In this way one obtains highly uniform effective bounds for results that are more general than the initial ones. While the methods used to obtain these new results come from mathematical logic, their proofs can be written in ordinary mathematics. We refer to Kohlenbach's book [18] for a comprehensive reference for proof mining.

This line of research, developed by Kohlenbach in the 1990s, has its origins in Kreisel's program of *unwinding of proofs*. Kreisel's idea was to apply proof-theoretic techniques to analyze concrete mathematical proofs and unwind the information hidden in them; see for example [22] and, more recently, [27].

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Proof mining has numerous applications to approximation theory, asymptotic behavior of nonlinear iterations, as well as (nonlinear) ergodic theory, topological dynamics and Ramsey theory. In these applications, Kohlenbach's *monotone* functional interpretation [14] is crucially used, since it systematically transforms any statement in a given proof into a new version for which explicit bounds are provided.

Terence Tao [37] arrived at a proposal of so-called *hard analysis* (as opposed to *soft analysis*), inspired by the finitary arguments used by him and Green [10] in their proof that there are arithmetic progressions of arbitrary length in the prime numbers, as well as by him alone in a series of papers [36,38–40]. As Kohlenbach points out in [17], Tao's hard analysis could be viewed as carrying out, using monotone functional interpretation, analysis on the level of uniform bounds.

For mathematical proofs based on classical logic, general logical metatheorems were obtained by Kohlenbach [16] for important classes of metrically bounded spaces in functional analysis and generalized to the unbounded case by Gerhardy and Kohlenbach [8]. They considered metric, hyperbolic and CAT(0)-spaces, (uniformly convex) normed spaces and inner product spaces also with abstract convex subsets. The metatheorems were adapted to Gromov δ-hyperbolic spaces and $\mathbb{R}$-trees [23], complete metric and normed spaces [18] and uniformly smooth Banach spaces [19]. The proofs of the metatheorems are based on extensions to the new formal systems of Gödel's functional interpretation combined with negative translation and parametrized versions of majorization. These logical metatheorems guarantee that one can extract effective uniform bounds from classical proofs of $\forall\exists$-sentences and that these bounds are independent from parameters satisfying weak local boundedness conditions. Thus, the metatheorems can be used to conclude the existence of effective uniform bounds without having to carry out the proof analysis: we have to verify only that the statement has the right logical form and that the proof can be formalized in our system.

Gerhardy and Kohlenbach [7] obtained similar logical metatheorems for proofs in semi-intuitionistic systems, that is proofs based on intuitionistic logic enriched with noneffective principles, such as comprehension in all types for arbitrary negated or $\exists$-free formulas. The proofs of these metatheorems use monotone modified realizability, a monotone version of Kreisel's modified realizability [21]. A great benefit of this setting is that there are basically no restrictions on the logical complexity of mathematical theorems for which bounds can be extracted.

The goal of this paper is to present an application of proof mining to the asymptotic behavior of Ishikawa iterations for nonexpansive mappings.

Let X be a normed space, $C \subseteq X$ a convex subset and $T : C \to C$. We shall denote with $Fix(T)$ the set of fixed points of T. The *Ishikawa iteration* starting with $x \in C$ was introduced in [13] as follows:

$$x_0 = x, \qquad x_{n+1} = (1-\lambda_n)x_n + \lambda_n T\big((1-s_n)x_n + s_n T x_n\big),$$

where $(\lambda_n), (s_n)$ are sequences in $[0,1]$. The well-known Krasnoselski–Mann iteration [20,28] is obtained as a special case by taking $s_n = 0$ for all $n \in \mathbb{N}$.

Ishikawa proved that for convex compact subsets C of Hilbert spaces and Lipschitzian pseudocontractive mappings T, this iteration converges strongly towards a fixed point of T, provided that the sequences (λ_n) and (s_n) satisfy some assumptions.

In the following we consider the Ishikawa iteration for nonexpansive mappings and sequences (λ_n), (s_n) satisfying the following conditions:

$$\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n) \quad \text{diverges}, \qquad \limsup_{n\to\infty} s_n < 1 \quad \text{and} \quad \sum_{n=0}^{\infty} s_n(1-\lambda_n) \quad \text{converges}. \tag{1}$$

Tan and Xu [35] proved the weak convergence of the Ishikawa iteration in uniformly convex Banach spaces X which satisfy Opial's condition or whose norm is Fréchet differentiable, generalizing in this way a well-