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Thin equivalence relations and inner models

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ABSTRACT

We describe the inner models with representatives in all equivalence classes of thin equivalence relations in a given projective pointclass of even level assuming projective determinacy. The main result shows that these models are characterized by their correctness and the property that they correctly compute the tree from the appropriate scale. The main step towards this characterization shows that the tree from a scale can be reconstructed in a generic extension of an iterate of a mouse. We then construct models with this property as generic extensions of iterates of mice under the assumption that the corresponding projective ordinal is below ω_2. On the way, we consider several related problems, including the question when forcing does not add equivalence classes to thin projective equivalence relations. For instance, we show that if every set has a sharp, then reasonable forcing does not add equivalence classes to thin provably Δ^1_3 equivalence relations, and generalize this to all projective levels.

1. Introduction

Definable equivalence relations are a focus of modern descriptive set theory, and while the bulk of research centers around Borel equivalence relations, there has been a large amount of work on projective equivalence relations, for example Harrington and Sami [13], Hjorth [15,17,18], Hjorth and Kechris [19], Kechris [24], Louveau and Rosendal [29], and Silver [42], and equivalence relations in the constructible universe $L(\mathbb{R})$ over the reals, for example Hjorth [16]. Inner model theory has proved to be very useful for this endeavor, in particular iterable models with Woodin cardinals (see e.g. Hjorth [17]). For the theory of iterable mice with Woodin cardinals see Mitchell and Steel [30] and Steel [43,46]. Earlier approaches as in Harrington and Sami [13] and Kechris [24] use direct proofs from determinacy. It is well known that determinacy and the existence of the appropriate mice are equivalent (see [7,28,32,35]).

This paper is about thin projective equivalence relations, i.e. those with no perfect set of pairwise inequivalent reals. They have been most notably studied by Harrington and Sami [13], motivated by the question

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about the number of equivalence classes. A starting point in this topic is Silver's theorem [42] that every thin $\boldsymbol{\Pi}^1_1$ equivalence relation has countably many equivalence classes. Harrington and Sami subsequently extended this through the projective hierarchy relative to the projective ordinals. The nth projective ordinal $\underset{\sim}{\delta}^1_n$ is the supremum of lengths of $\boldsymbol{\Delta}^1_n$ prewellorders. The number of equivalence classes of thin $\boldsymbol{\Pi}^1_n$ equivalence relations is below $\underset{\sim}{\delta}^1_n$ if $n \geq 1$ is odd and at most the size of $\underset{\sim}{\delta}^1_{n-1}$ if $n \geq 2$ is even [13].

A quite different approach to this question is to look for a bound for the number of equivalence classes of co-κ-Suslin equivalence relations, i.e. when the complement is the projection of a tree T on $\omega \times \omega \times \kappa$. Harrington and Shelah [14] showed this is at most κ if the complement of $p[T]$ is an equivalence relation in a Cohen generic extension. We use this to bound the number of equivalence classes of thin $\boldsymbol{\Pi}^1_n$ equivalence relations under the assumption that the pointclasses $\boldsymbol{\Pi}^1_k$ are scaled for odd k and all projective sets have the Baire property. Note that scales are closely connected to Suslin representations.

Since the number of equivalence classes of thin $\boldsymbol{\Pi}^1_n$ equivalence relations is bounded by a projective ordinal, we look for inner models (possibly with fewer reals than V) which have representatives in all equivalence classes of all thin $\Pi^1_n(x)$ equivalence relations, where x is a real parameter in the inner model. Hjorth [15] showed that every inner model has this property for $n = 1$ as a consequence of Silver's theorem. The candidates for such inner models for $n \geq 2$ are forcing extensions of fine structural inner models with Woodin cardinals. We will construct such models if the corresponding projective ordinal is below ω_2.

Hjorth [15] characterized the models with this property for $n = 2$. Assuming all reals have sharps, the inner models with this property for $n = 2$ are exactly the Σ^1_3 correct inner models with the right ω_1. We extend Hjorth's theorem to the even levels in the projective hierarchy in Main Theorem 5.15. The level of correctness is adapted and instead of asking that the model has the right ω_1, we ask that the model correctly computes the tree T_{2n+1} from the canonical Π^1_{2n+1}-scale, and assume the appropriate amount of determinacy. Thus these inner models are characterized in a simple and beautiful way.

The proof generalizes Hjorth's proof. In the harder direction, Main Lemma 5.2 shows that the tree T^M_{2n+1} from the canonical Π^1_{2n+1}-scale as computed in an inner model M with countably many reals can be reconstructed in an iterate of $M^\#_{2n}$. To do this, Woodin's genericity iteration is applied to make reals generic at local Woodin cardinals over iterates of $M^\#_{2n}$, and we force over the direct limit. A density argument will show that the tree can be defined. We also have to look more closely at the Harrington–Shelah result [14]. If the equivalence relation is co-κ-Suslin, then for any real there is an infinitary formula defining a neighborhood inside its equivalence class. Combining this with Steel's result that $M^\#_n$ is coded by a projective real, we can express the existence of a real in this neighborhood in a projective way and use this to complete the proof.

Let's look at the setting from a different perspective and suppose the universe is a forcing extension of an inner model. The issue is when a forcing introduces new equivalence classes to thin projective equivalence relations. Foreman and Magidor [9] showed that reasonable forcing of size at most κ does not add equivalence classes to thin κ-weakly homogeneously Suslin equivalence relations. Together with the Martin–Steel theorem [8] this implies that if there are infinitely many Woodin cardinals, then reasonable forcing does not add equivalence classes to thin projective equivalence relations. We replace the large cardinal assumption with the weaker assumption that $M^\#_n(X)$ exists for all $X \in H_{\kappa^+}$ and show, using the absoluteness of $M^\#_n(X)$, that reasonable forcing of size at most κ does not add new equivalence classes to thin provably $\boldsymbol{\Delta}^1_{n+2}$ equivalence relations.

The paper is organized as follows. Section 1 introduces the facts about thin equivalence relations, prewellorders, scales, and properties of $M^\#_n$ which we will use.

In Section 2 we study liftings of thin projective equivalence relations to forcing extensions. We show based on an idea of Foreman and Magidor [9] that for any infinite cardinal κ, reasonable forcing of size at most κ does not introduce new equivalence classes to thin projective equivalence relations if $M^\#_n(X)$ exists for every self-wellordered set $X \in H_{\kappa^+}$ and every n. The argument is adapted to $\boldsymbol{\Sigma}^1_2$ c.c.c. forcings. We show that generic Σ^1_{n+3} absoluteness holds for these forcings from the assumption that $M^\#_n(x)$ exists for