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Cascades, order, and ultrafilters

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ABSTRACT

We investigate mutual behavior of cascades, contours of which are contained in a fixed ultrafilter. This allows us to prove (ZFC) that the class of strict J_{ω^ω}-ultrafilters, introduced by J.E. Baumgartner in [2], is empty. We translate the result to the language of $<_\infty$-sequences under an ultrafilter, investigated by C. Laflamme in [17], and we show that if there is an arbitrary long finite $<_\infty$-sequence under u, then u is at least a strict $J_{\omega^\omega+1}$-ultrafilter.

1. Introduction

J.E. Baumgartner in [2] introduced a notion of $\mathcal{I}$-ultrafilter. Let $\mathcal{I}$ be an ideal on X. An ultrafilter u on ω is called an $\mathcal{I}$-ultrafilter if for every function $f : \omega \to X$ there is a set $U \in u$ such that $f[U] \in \mathcal{I}$. Baumgartner's $\mathcal{I}$-ultrafilters were studied by several mathematicians. Here we mention only the papers which are most important from our point of view: C. Barney [1], A. Błaszczyk [3], J. Brendle [4], J. Flašková [9–11], C. Laflamme [17], S. Shelah [18,19].

Among $\mathcal{I}$-ultrafilters Baumgartner investigated ordinal ultrafilters, hierarchized in an ω_1-sequence of classes of ultrafilters. We say that u is J_α ultrafilter (on ω) if for each function $f : \omega \to \omega_1$ there is $U \in u$ such that $\mathrm{ot}(f[U]) < \alpha$, where $\mathrm{ot}(\cdot)$ denotes the order type. An ultrafilter u on ω is a strict J_α-ultrafilter if u is a J_α-ultrafilter and u is not J_β-ultrafilter for any $\beta < \alpha$. For additional information about ordinal ultrafilters a look at [2,4,21,22] is recommended.

In [2] J.E. Baumgartner proved (in Theorems 4.2 and 4.6) that for each successor ordinal $\alpha < \omega_1$ the class of strict J_{ω^α}-ultrafilters is nonempty if P-points exist. He also pointed out that the following problem is open even if CH or MA is assumed.

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Problem 1.1. If α is a limit ordinal, is there a strict-J_{ω^α} ultrafilter?

In the present paper we partially solve the problem, showing (ZFC) that the class of strict J_{ω^ω}-ultrafilters is empty.

If u is a filter(-base) on $A \subset B$, then we identify u with the filter on B for which u is a filter-base. Given ultrafilters u, v on ω, recall that $v <_\infty u$ if there is a function $f : \omega \to \omega$ such that $f(u) = v$ and f is neither finite-to-one nor constant on each set $U \in u$. Laflamme proved that if an ultrafilter u has an infinite decreasing $<_\infty$-sequence below, then u is at least strict $J_{\omega^{\omega+1}}$-ultrafilter (cf. [17, Lemma 3.2]). He also stated the following

Problem 1.2. What about the corresponding influence of increasing $<_\infty$-chains below u? Given such an ultrafilter u with an increasing infinite $<_\infty$-sequence $u >_{RK} \ldots >_\infty u_1 >_\infty u_0$ below, fix maps g_i and f_i witnessing $u >_{RK} u_i$ and $u_{i+1} >_\infty u_i$ respectively. The problem is really about the possible connections between g_i and $f_i \circ g_{i+1}$ even relative to members of u.

Problem 1.3. Can we have an ultrafilter u with arbitrary long finite $<_\infty$-chains below u without infinite one? This looks like the most promising way to build a strict J_{ω^ω}-ultrafilter.

We find affirmative answer to Problem 1.2 and negative answer Problem 1.3.

2. Preliminaries

In [7] S. Dolecki and F. Mynard introduced monotone sequential cascades, special kind of well-founded trees, as a tool to describe topological sequential spaces. Cascades and their contours proved to be useful in the investigation of certain types of ultrafilters on ω, namely P-hierarchy (see [21,22]) and ordinal ultrafilters. Here we focus on ordinal ultrafilters.

The *cascade* is a tree V, ordered by "$\sqsubseteq$", without infinite branches and with the least element $\emptyset_V$. A cascade is *sequential* if for each non-maximal element v of V ($v \in V \setminus \max V$) the set v^{+V} of immediate successors of v (in V) is countably infinite. We say that v is a *predecessor* of v' (we write $v = \mathrm{pred}(v')$) if $v' \in v^{+V}$. We write v^+ instead of v^{+W} if it is known in which cascade the successors of v are considered. As a convention, since $\max V$ is infinitely countable, we think about $\max V$ as of a copy (or of a subset) of ω.

The *rank* of $v \in V$ ($r_V(v)$ or $r(v)$) is defined inductively as follows: $r(v) = 0$ if $v \in \max V$, and otherwise $r(v)$ is the least ordinal greater than the ranks of all immediate successors of v. The rank $r(V)$ of the cascade V is, by definition, the rank of $\emptyset_V$.

The cascade V is said to be *monotone* if it is possible to order all sets v^+ (for $v \in V \setminus \max V$) obtaining sequences $(v_n)_{n<\omega}$ so that for each $v \in V \setminus \max V$ the sequence $(r(v_n)_{n<\omega})$ is non-decreasing, and we fix such an order on V without indication. In other words if for each $v \in V \setminus \emptyset_V$ the set $\{v' \in (\mathrm{pred}(v))^+ : r(v') < r(v)\}$ is finite. We then can introduce the lexicographic order $<_{lex}$ on V in the following way: $v' <_{lex} v''$ if $v' \sqsupset v''$ or if there exist $\widetilde{v'} \sqsubseteq v'$, $\widetilde{v''} \sqsubseteq v''$ and v such that $\widetilde{v'} \in v^+$ and $\widetilde{v''} \in v^+$ and $\widetilde{v'} = v_n$, $\widetilde{v''} = v_m$ and $n < m$.

Let W be a cascade, and let $\{V^w : w \in \max W\}$ be a set of pairwise disjoint cascades such that $V^w \cap W = \emptyset$ for each $w \in \max W$. The *confluence* of cascades V^w with respect to the cascade W (we write $W \looparrowleft V^w$) is defined as a cascade constructed by the identification of $w \in \max W$ with $\emptyset_{V^w}$ and according to the following rules: (1) $\emptyset_W = \emptyset_{W \looparrowleft V^w}$; (2) if $w \in W \setminus \max W$, then $w^{+W \looparrowleft V^w} = w^{+W}$; (3) if $w \in V^{w_0}$, for a certain $w_0 \in \max W$, then $w^{+W \looparrowleft V^w} = w^{+V^{w_0}}$; (4) in each case we also assume that the order on the set of successors remains unchanged. By $(n) \looparrowleft V^n$ we denote $W \looparrowleft V^w$ where W is a sequential cascade of rank 1.

Also we label elements of a cascade V by finite sequences of natural numbers of length $r(V)$ or less, by the function f which preserves the lexicographic order, $v_{f(v)}$ is the resulting name for an element of