Annals of Pure and Applied Logic 165 (2014) 1291–1300

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

A reducibility related to being hyperimmune-free ☆

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ARTICLE INFO

Article history:
Received 10 July 2013
Received in revised form 25 March 2014
Accepted 8 April 2014
Available online 5 May 2014

MSC:
primary 03D30
secondary 03D28

Keywords:
Recursion theory
Reducibilities
Hyperimmune
Turing degrees
Majorisation

ABSTRACT

The main topic of the present work is the relation that a set X is strongly hyperimmune-free relative to Y. Here X is strongly hyperimmune-free relative to Y if and only if for every partial X-recursive function p there is a partial Y-recursive function q such that every a in the domain of p is also in the domain of q and satisfies $p(a) < p(a)$, that is, p is majorised by q. For X being hyperimmune-free relative to Y, one also says that X is pmaj-reducible to Y ($X \leq_{\mathrm{pmaj}} Y$). It is shown that between degrees not above the Halting problem the pmaj-reducibility coincides with the Turing reducibility and that therefore only recursive sets have a strongly hyperimmune-free Turing degree while those sets Turing above the Halting problem bound uncountably many sets with respect to the pmaj-relation. So pmaj-reduction is identical with Turing reduction on a class of sets of measure 1. Moreover, every pmaj-degree coincides with the corresponding Turing degree. The pmaj-degree of the Halting problem is definable with the pmaj-relation.

1. Introduction

Post [7] introduced the notions of immune and hyperimmune sets in order to search for conditions on the complements of r.e. sets which guarantee incompleteness for certain reducibilities. In the subsequent study [4–6,8,10,11,14,15] the notion of hyperimmunity played a central role and also the discovery that there are Turing degrees which do not contain a hyperimmune set, these are called the hyperimmune-free Turing degrees. This was generalised by saying that X is hyperimmune-free relative to Y if every total X-recursive function is majorised by a total Y-recursive function. One could generalise this notion as follows, where one says that a partial function q dominates a partial function p iff for almost all x in the domain of p it holds that x is also in the domain of q and $p(x) < q(x)$; furthermore, q majorises p if the domain of p is a subset

☆ F. Stephan is supported in part by NUS grant R252-000-420-112. L. Yu is partially supported by NSF of China grant No. 11322112. Both the authors thank Wolfgang Merkle for his helpful discussion and his hospitality when visiting Heidelberg.

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http://dx.doi.org/10.1016/j.apal.2014.04.002

of the domain of q and $p(x) < q(x)$ for all x in the domain of p. Now, the following six notions can arise in principle:

1d: One total Y-recursive function dominates every total X-recursive function;
1m: Every total X-recursive function is majorised by a total Y-recursive function;
2d: One total Y-recursive function dominates every partial X-recursive function;
2m: Every partial X-recursive function is majorised by a total Y-recursive function;
3d: One partial Y-recursive function dominates every partial X-recursive function;
3m: Every partial X-recursive function is majorised by a partial Y-recursive function.

Notion **1m** is closely related to the above discussed notion of hyperimmune-freeness. Notion **1d** has a strong resemblance to the low-high hierarchy: If $X \leq_T Y$ then Y dominates X in sense **1d** iff $X'' \leq_T Y'$ by a result of Martin; in particular fixing X as recursive would give that the Y ranges over the high degrees while fixing $Y = K$ and taking X to be Δ_2^0 would imply that X is dominated in sense **1d** by Y iff X is low$_2$. The notions **2d**, **2m** and **3d** all coincide and are the strongest form of domination which one can get and imply that $X' \leq_T X \oplus Y$. Notion **3m** is the notion of pmaj-reducibility which will be discussed in the following.

So, X is strongly hyperimmune-free relative to Y iff every partial X-recursive function is majorised by a partial Y-recursive function. In contrast to hyperimmune-free degrees, it turns out that no nonrecursive Turing degree is strongly hyperimmune-free. So the more interesting part is the overall relation between sets X and Y than the special case $X \leq_{\text{pmaj}} \emptyset$.

Concerning the downward closure the following result is obtained: If $X \not\geq_T K$ then $\{Y : Y \leq_{\text{pmaj}} X\} = \{Y : Y \leq_T X\}$ else $\{Y : Y \leq_{\text{pmaj}} X\}$ is uncountable and has measure 1. In particular, $\{Y : Y \leq_{\text{pmaj}} K\}$ contains all sets which are Martin-Löf random relative to K, all sets which are low for Ω, which are jump traceable and which are Turing reducible to K.

Furthermore, if a single function q majorises all partial-recursive functions then the Turing degree of q is at least K; hence K is the first dominant degree for the pmaj-degrees. This stands in contrast to Chong's notion of pdomination: He defined that a set X is pdominant iff there is a single partial X-recursive function q such that for all partial-recursive functions p and almost all x in the domain of p there is a $y \leq x$ in the domain of q with $p(x) < q(y)$. This type of domination is easier to obtain and Chong, Hoi, Stephan and Turetsky [1] study in detail which degrees are pdominant in this sense: for example some low r.e. degrees are pdominant while other high r.e. degrees fail to be pdominant. The notion of pdominance is quite orthogonal to many known recursion-theoretic classes of oracles.

Although Shore and Slaman [13] prove that the Turing degree of Halting problem is definable and subsequently Shore [12] found a more direct definition, however both the definitions are quite sophisticated. The pmaj-reducibility can be viewed as a natural and slight modification of the Turing reducibility (in the sense of measure theory, they are identical almost everywhere). With this reduction, one may prove that the Turing degree of Halting problem can be defined in a significantly simpler way.

2. Basic facts

Now the formal definition of when a set X is strongly hyperimmune-free relative to another set Y is given.

Definition 2.1. X is strongly hyperimmune-free relative to Y, written $X \leq_{\text{pmaj}} Y$, if for every partial X-recursive function p there is a partial Y-recursive function q which majorises p, that is, which satisfies for all $n \in dom(p)$ that $n \in dom(q)$ and $p(n) < q(n)$.