Annals of Pure and Applied Logic 165 (2014) 1301–1317

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Computable categoricity for pseudo-exponential fields of size $\aleph_1$ ☆

Jesse Johnson

Westfield State University, Wilson Hall 409, Westfield, MA 01086, United States

A R T I C L E   I N F O

Article history:
Received 3 December 2012
Received in revised form 14 March 2014
Accepted 9 April 2014
Available online 26 April 2014

MSC:
03C57
03D60
03C70
03C52

Keywords:
Pseudo-exponentiation
Computable categoricity

A B S T R A C T

We use some notions from computability in an uncountable setting to describe a difference between the "Zilber field" of size $\aleph_1$ and the "Zilber cover" of size $\aleph_1$.

1. Introduction

We begin with some definitions necessary for analyzing computable categoricity of "pseudo-exponential fields" and other structures of size $\aleph_1$. The pseudo-exponential fields are structures that resemble the field of complex numbers with complex exponentiation. Zilber listed five axioms in $L_{\omega_1,\omega}(Q)$, which characterize the class of pseudo-exponential fields, and he showed (with added work in a new paper of Bays et al. [2]) that there is a unique model in each uncountable power [16]. Zilber further conjectured that the pseudo-exponential field of size $2^{\aleph_0}$ is the complex exponential field $\mathbb{C}_{exp}$. A related class of structures, the "Zilber covers," is axiomatized using $L_{\omega_1,\omega}$, without Q. Zilber showed that there is a unique cover in each uncountable power [17]. The one of cardinality $2^{\aleph_0}$ is the cover derived from $(\mathbb{C}, exp)$. In this paper, we show that the pseudo-exponential field of size $\aleph_1$ is not computably categorical, while the cover is relatively computably categorical.

We will be working with structures of size $\aleph_1$, and we shall use notions of computability appropriate for these structures. The notions of computability are those of α-recursion theory, where $\alpha = \omega_1$. For an

☆ This work was supported by a Research Training Groups grant (EMSW21-RTG-0838506) from the National Science Foundation.
E-mail address: jwjohnson@westfield.ma.edu.

http://dx.doi.org/10.1016/j.apal.2014.04.003

introduction to computable structure theory in this setting, see Greenberg and Knight [6] or Johnson [8]. For some nice results on computable categoricity and linear orders in the setting of ω_1-computability, see [7].

We begin with two simpler examples: classes with a unique member in each uncountable power, one axiomatizable using $L_{\omega_1,\omega}(Q)$, and the other using just $L_{\omega_1,\omega}$. The first is relatively computably-categorical while the second will be not even computably categorical. We will then describe the pseudo-exponential fields, pointing out the properties that we use in our proofs. We also describe the Zilber covers. We locate the property that accounts for the difference in computable categoricity. This is related to the fact that the pseudo-exponential fields require the use of the second-order quantifier Q, while the covers are axiomatized by an $L_{\omega_1,\omega}$-sentence.

1.1. Basic definitions from α-recursion for $\alpha = \omega_1$

The universal domain for ω_1-computability is L_{ω_1}. We assume that $V = L$. We assume this because first, it implies that the reals are all present in L_{ω_1}, and second, it will be important that for all $A \subseteq L_{\omega_1}$, L_{ω_1} is closed under the function $x \mapsto A \cap x$. We can enrich the language of set theory, naming each element of L_{ω_1}. We will use $\Sigma_1(L_{\omega_1})$ to denote the collection of finitary formulas with bounded and existential quantifiers in the language of set theory, with the added constants.

Definition 1.

- An n-ary relation $R \subseteq (L_{\omega_1})^n$ is *bounded*, or Δ_0, if it is defined in $(L_{\omega_1}, \in)$ by a formula, with parameters, built from atomic sub-formulas using Boolean combinations and bounded quantifiers $(\exists x \in y)$ and $(\forall x \in y)$.
- A relation $R \subseteq (L_{\omega_1})^n$ is *computably enumerable*, or *c.e.*, if it is defined by a $\Sigma_1(L_{\omega_1})$-formula, where this is a formula with bounded and existential quantifiers in front of a quantifier-free formula.
- A relation $R \subseteq (L_{\omega_1})^n$ is *computable* if it and its complement are computably enumerable.
- A partial function $f : (L_{\omega_1})^n \to L_{\omega_1}$ is *partial computable* if its graph is a c.e. relation.
- A *computable function* $f : (L_{\omega_1}) \to L_{\omega_1}$ is a partial computable function whose domain is computable.

Using results of Gödel, we get a computable 1-1 function g from the countable ordinals onto L_{ω_1} where the relation $g(\alpha) \in g(\beta)$ is computable. So, the function g gives us ordinal codes for sets in L_{ω_1}, where α is a code for $g(\alpha)$. Furthermore, there is a computable function ℓ that takes α to the code for L_α. Thus, computing in ω_1 is essentially the same as computing in L_{ω_1}. We will use searches over the L_α's for $\alpha < \omega_1$. For more details, see [5] or [6]. Technically, we should use the term "ω_1-computable" to indicate that we are computing over the universe ω_1, or L_{ω_1}, instead of ω. However, for simplicity, it will be understood that for the present paper "computable" indicates ω_1-computable.

As in the standard setting of computability, we have indices for c.e. sets. We have a c.e. set C of codes for pairs $(\varphi, \bar{c})$, each representing a Σ_1 definition, where $\varphi(\bar{u}, x)$ is a Σ_1 formula and $\bar{c}$ is a tuple of parameters appropriate for $\bar{u}$. We also have a computable function h that maps ω_1 onto C. We say that $\alpha < \omega_1$ is a *c.e. index* for A if $h(\alpha)$ is the code for a pair $(\varphi, \bar{c})$, where $\varphi(\bar{c}, x)$ is a Σ_1 definition of A in $(L_{\omega_1}, \in)$.

Definition 2.

- We will write W_α for the c.e. set defined by the pair $h(\alpha)$ in C, where $h(\alpha) = (\varphi, \bar{c})$.
- We write $x \in W_{\alpha,\beta}$ if L_β contains x, the parameters $\bar{c}$, and the witnesses making the formula $\varphi(\bar{c}, x)$ true.

  Note that the relation $x \in W_{\alpha,\beta}$ is computable; we say $x \notin W_{\alpha,\beta}$ if $(L_\beta, \in) \neg\varphi(\bar{c}, x)$, which involves only bounded quantifiers.