Annals of Pure and Applied Logic 164 (2013) 1236–1246

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Uniformly defining valuation rings in Henselian valued fields with finite or pseudo-finite residue fields

Raf Cluckers [a,b], Jamshid Derakhshan [c,*], Eva Leenknegt [d], Angus Macintyre [e]

[a] *Université Lille 1, Laboratoire Painlevé, CNRS – UMR 8524, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex, France*
[b] *Katholieke Universiteit Leuven, Department of Mathematics, Celestijnenlaan 200B, B-3001 Leuven, Belgium*
[c] *University of Oxford, Mathematical Institute, 24-29 St Giles', Oxford OX1 3LB, UK*
[d] *Purdue University, Department of Mathematics, 150 N. University Street, West Lafayette, IN 47907-2067, US*
[e] *School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK*

A R T I C L E I N F O

Article history:
Received 15 December 2011
Received in revised form 17 October 2012
Accepted 17 October 2012
Available online 22 July 2013

MSC:
primary 11D88, 11U09
secondary 11U05, 03C60

Keywords:
Definability
Diophantine sets
Hilbert's tenth problem

A B S T R A C T

We give a definition, in the ring language, of $\mathbb{Z}_p$ inside $\mathbb{Q}_p$ and of $\mathbb{F}_p[[t]]$ inside $\mathbb{F}_p((t))$, which works uniformly for all p and all finite field extensions of these fields, and in many other Henselian valued fields as well. The formula can be taken existential-universal in the ring language, and in fact existential in a modification of the language of Macintyre. Furthermore, we show the negative result that in the language of rings there does not exist a uniform definition by an existential formula and neither by a universal formula for the valuation rings of all the finite extensions of a given Henselian valued field. We also show that there is no existential formula of the ring language defining $\mathbb{Z}_p$ inside $\mathbb{Q}_p$ uniformly for all p. For any fixed finite extension of $\mathbb{Q}_p$, we give an existential formula and a universal formula in the ring language which define the valuation ring.

1. Introduction

Uniform definitions of valuation rings inside families of Henselian valued fields have played important roles in the work related to Hilbert's 10th problem by B. Poonen [11] and by J. Koenigsmann [8], especially uniformly in p-adic fields. We address this issue in a wider setting, using the ring language and Macintyre's language. Since the work [9], the Macintyre language has always been prominent in the study of p-adic fields.

* Corresponding author.
E-mail addresses: Raf.Cluckers@math.univ-lille1.fr (R. Cluckers), derakhsh@maths.ox.ac.uk (J. Derakhshan), eleenkne@math.purdue.edu (E. Leenknegt), angus@eecs.qmul.ac.uk (A. Macintyre).
URL: http://math.univ-lille1.fr/~cluckers (R. Cluckers).

0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2013.06.010

Let $\mathcal{L}_{\text{ring}}$ be the ring language $(+, -, \cdot, 0, 1)$. Write $\mathcal{L}_{\text{Mac}}$ for the language of Macintyre, which is obtained from $\mathcal{L}_{\text{ring}}$ by adding for each integer $n > 0$ a predicate P_n for the set of nonzero n-th powers. We assume that the reader is familiar with pseudo-finite fields and Henselian valued fields. For more information we refer to [5], [10], [4], and [3].

The following notational conventions are followed in this paper. For a Henselian valued field K we will write $\mathcal{O}_K$ for its valuation ring. $\mathcal{O}_K$ is assumed nontrivial. $\mathcal{M}_K$ is the maximal ideal of $\mathcal{O}_K$, and $k = \mathcal{O}_K/\mathcal{M}_K$ is the residue field. We denote by res the natural map $\mathcal{O}_K \to k$, and by ord the valuation.

Given a ring R and a formula φ in $\mathcal{L}_{\text{ring}}$ or $\mathcal{L}_{\text{Mac}}$ in $m \geqslant 0$ free variables, we write $\varphi(R)$ for the subset of R^m consisting of the elements that satisfy φ. In this paper we will always work without parameters, that is, with $\emptyset$-definability.

Theorem 1. *There is an existential formula $\varphi(x)$ in $\mathcal{L}_{\text{ring}} \cup \{P_2, P_3\}$ such that*

$$\mathcal{O}_K = \varphi(K)$$

holds for any Henselian valued field K with finite or pseudo-finite residue field k provided that k contains non-cubes in case its characteristic is 2.

We are very grateful to an anonymous referee for pointing out to us that our argument in an earlier version failed when k has characteristic 2 and every element is a cube (i.e. $(k^*)^3 = k^*$). There are such k, finite ones and pseudo-finite ones (cf. Appendix A).

Note that in such a case k has no primitive cube root of unity, and so its unique quadratic extension is cyclotomic. That extension is the Artin–Schreier extension, and (as the referee suggested) it is appropriate to adjust the Macintyre language by replacing P_2 by P_2^{AS}, where

$$P_2^{AS}(x) \quad \Leftrightarrow \quad \exists y \ \bigl(x = y^2 + y\bigr).$$

This has notable advantages, namely:

Theorem 2. *There is an existential formula $\varphi(x)$ in $\mathcal{L}_{\text{ring}} \cup \{P_2^{AS}\}$ such that*

$$\mathcal{O}_K = \varphi(K)$$

holds for all Henselian valued fields K with finite or pseudo-finite residue field.

Since in a field of characteristic not equal to 2, we have $P_2^{AS}(x) \Leftrightarrow P_2(1 + 4x)$, Theorem 2 implies the following.

Theorem 3. *There is an existential formula $\varphi(x)$ in $\mathcal{L}_{\text{ring}} \cup \{P_2\}$ such that*

$$\mathcal{O}_K = \varphi(K)$$

holds for all Henselian valued fields K with finite or pseudo-finite residue field of characteristic not equal to 2.

Before proving the above theorems, we state some other results. First some negative results.

Theorem 4. *Let K be any Henselian valued field. There does not exist an existential formula $\psi(x)$ in $\mathcal{L}_{\text{ring}}$ such that*