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Proof-theoretic conservations of weak weak intuitionistic constructive set theories

Lev Gordeev [a,b,*]

[a] *Universität Tübingen, Tübingen, Germany*
[b] *Ghent University, Ghent, Belgium*

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Dedicated to the memory of Professor N.A. Shanin

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ABSTRACT

The paper aims to provide precise proof theoretic characterizations of Myhill–Friedman-style "weak" constructive extensional set theories and Aczel–Rathjen analogous constructive set theories both enriched by Mostowski-style collapsing axioms and/or related anti-foundation axioms. The main results include full intuitionistic conservations over the corresponding purely arithmetical formalisms that are well known in the reverse mathematics – which strengthens analogous results obtained by the author in the 80s. The present research was inspired by the more recent Sato-style "weak weak" classical extensional set theories whose proof theoretic strengths are shown to strongly exceed the ones of the intuitionistic counterparts in the presence of the collapsing axioms.

1. Introduction

J. Myhill [6] and H. Friedman [11] introduced several constructively meaningful principles and formal systems of *weak* extensional intuitionistic set theory whose proof-theoretic strengths was shown [11] to range between that of standard first- and second-order Arithmetic, **PA** (or **HA**) and $\mathbf{PA_2}$ (or $\mathbf{HA_2}$), respectively, thus being essentially weaker than standard classical set theory **ZF**. Furthermore [11] posed a deeper problem conjecturing that intuitionistic set theories under consideration are conservative extensions of the underlying arithmetical intuitionistic formalisms. These conjectures (et al.) have been confirmed [14] for Friedman's extensional set theories $\mathbf{T_1}$, $\mathbf{T_2}$, $\mathbf{T_3}$ having proof-theoretic strengths $|\mathbf{T_1}| = \varepsilon_0$, $|\mathbf{T_2}| = \varphi_{\varepsilon_0}(0)$,

* Corresponding author at: Universität Tübingen, Tübingen, Germany.
E-mail addresses: lew.gordeew@uni-tuebingen.de, lew.gordeew@ugent.be.

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$|\mathbf{T_3}| = \varphi_{\varepsilon_{\Omega+1}}(0)$ (also known as Howard ordinal $|\mathbf{ID}_1|$). Moreover [14] strengthened Friedman's conjectures by also proving conservations in the presence of consistent combinations of other constructive principles like an anti-foundation axiom **Cpl** and/or finite-types axiom of choice $\mathbf{AC}_{\text{FT}}$ [14, Corollary 3, p. 35]. Actually for every weak set theory **S** in question [14] expressed the solution in the "most conservative" form $\mathbf{S} \vdash A \Leftrightarrow \mathbf{HA} + \mathbf{TI}\ (< |\mathbf{S}|) \vdash A$, for any arithmetical statement A, where $\mathbf{TI}\ (< |\mathbf{S}|)$ denotes the arithmetical transfinite induction scheme below proof theoretic ordinal of **S**. The proofs were based on the author's constructive semantics [13] of analogous weak set theories.

Working in classical logic K. Sato [19] introduced several weak refinements of basic weak set theory – both intensional and extensional – and determined their proof-theoretic ordinals. Notably Sato's *weak weak* classical set theories are less expressive than Myhill–Friedman's *weak* intuitionistic formalisms that can (arguably) mimic constructive essence of entire **ZF**. In particular, Zermelo's power set axiom

Pow: "*for every x there exists the set of all subsets of x*"

has natural constructive interpretation in the form

Exp: "*for every x and y there exists the set of all functions from x to y*"

occurring in Myhill–Friedman's theories. Since constructive functions are thought to simulate only algorithms, **Exp** is weaker than **Pow** in the intuitionistic environment. This might illuminate proof-theoretic weakness of Myhill–Friedman's intuitionistic formalisms, and on the other hand explain the lack of **Exp** in Sato's classical ones. It follows from the preceding that adding classical law of excluded middle to *weak* intuitionistic set theory with **Exp** like, say, $\mathbf{T_1}$, can raise its proof theoretic ordinal ε_0 far beyond that of, say, finite-order arithmetic $\mathbf{PA}_{<\omega}$. Note that, instead of **Exp**, Sato's basic set theory **Basic** includes Mostowski-style collapsing axiom

Clps: "*every well-founded relation r on a given ordinal x can be collapsed onto a transitive set y*"

and (unlike Friedman's $\mathbf{T}_i$) admits consistent extensions by Aczel-style anti-foundation axioms. It is thus natural to investigate proof-theoretic strengths of Sato's *weak weak* intuitionistic constructive set theories and following [19] ask whether they are conservative extensions of the underlying arithmetical intuitionistic formalisms. To address these questions we recall Sato's basic results

Theorem 1. *(See [19].)* $|\mathbf{Basic} + \mathbf{Ext}| = \varepsilon_0$, $|\mathbf{Basic} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep}| = \Gamma_0$.

and observe that the increase of proof theoretic strength of Δ_0-**Sep**, relative to $\mathbf{Basic} + \mathbf{Ext}$, is caused by essentially classical argument that allows to infer the comparability of countable well-orderings from **Basic**'s collapsing axiom **Clps**. This argument fails intuitionistically and we refine Sato's results by the following theorem, where $\mathbf{Basic}^{(\mathbf{i})}$ is the intuitionistic counterpart of **Basic**.

Theorem 2. $|\mathbf{Basic}^{(\mathbf{i})} + \mathbf{Ext}| = |\mathbf{Basic}^{(\mathbf{i})} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep} + \mathbf{Exp}| = \varepsilon_0$. *Moreover* $\mathbf{Basic}^{(\mathbf{i})} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep} + \mathbf{Exp}$ *is a conservative extension of* HA.

Thus, in particular, switching to classical logic in $\mathbf{Basic}^{(\mathbf{i})} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep}$ would raise proof theoretic strength from ε_0 up to Γ_0. Furthermore, strengthening $\mathbf{Basic}^{(\mathbf{i})} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep}$ by **Exp** still won't affect its proof theoretic strength (ε_0), whereas classical $\mathbf{Basic} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep} + \mathbf{Exp}$ is just as strong as $\mathbf{Basic} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep} + \mathbf{Pow}$ and hence $|\mathbf{Basic} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep} + \mathbf{Exp}| \geqslant |\mathbf{PA}_{<\omega}|$ (see above).

Related results about other/stronger set theoretic formalism are exposed in Chapter 5. The proof techniques are the same as those used in [13,14]. In the sequel we adopt basic notations and abbreviations used