Annals of Pure and Applied Logic 164 (2013) 1322–1337

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Chain conditions in dependent groups

Itay Kaplan [a,*,1], Saharon Shelah [b,c,2]

[a] Mathematisches Institut, Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany
[b] The Hebrew University of Jerusalem, Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, Jerusalem 91904, Israel
[c] Department of Mathematics, Hill Center-Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

A R T I C L E I N F O

Article history:
Received 6 December 2011
Received in revised form 19 July 2012
Accepted 19 July 2012
Available online 2 July 2013

MSC:
03C45
03C98
05D99

Keywords:
Dependent theories
NIP
Strongly dependent theories
Chain conditions
dp-Rank
Type definable groups

A B S T R A C T

In this note we prove and disprove some chain conditions in type definable and definable groups in dependent, strongly dependent and strongly^2 dependent theories.

1. Introduction

This note is about chain conditions in dependent, strongly dependent and strongly^2 dependent theories.

Throughout, all formulas will be first order, T will denote a complete first order theory, and $\mathfrak{C}$ will be the monster model of T—a very big saturated model that contains all small models. We do not differentiate between finite tuples and singletons unless we state it explicitly.

* Corresponding author.
E-mail address: itay.kaplan@uni-muenster.de (I. Kaplan).

[1] The first author's research was partially supported by the SFB 878 grant.
[2] The second author would like to thank the Israel Science Foundation for partial support of this research (Grant Nos. 710/07 and 1053/11). No. 993 on the author's list of publications.

0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2013.06.014

Definition 1.1. A formula $\varphi(x, y)$ has the independence property in some model if for every $n < \omega$ there are $\langle a_i, b_s \mid i < n,\ s \subseteq n \rangle$ such that $\varphi(a_i, b_s)$ holds iff $i \in s$.

A (first order) theory T is dependent (sometimes also NIP) if it does not have the independence property: there is no formula $\varphi(x, y)$ that has the independence property in any model of T. A model M is dependent if $Th(M)$ is.

A good introduction to dependent theories appears in [2], but we shall give an exact reference to any fact we use, so no prior knowledge is assumed.

What do we mean by a chain condition? Rather than giving an exact definition, we give an example of such a condition—the first one. It is the Baldwin–Saxl lemma, which we shall present with the (very easy and short) proof.

Definition 1.2. Suppose $\varphi(x, y)$ is a formula. Then if G is a definable group in some model, and for all $c \in C$, $\varphi(x, c)$ defines a subgroup, then $\{\varphi(\mathfrak{C}, c) \mid c \in C\}$ is a family of *uniformly definable subgroups*.

Lemma 1.3. *(See [3].) Let G be a group definable in a dependent theory. Suppose $\varphi(x, y)$ is a formula and that $\{\varphi(x, c) \mid c \in C\}$ defines a family of subgroups of G. Then there is a number $n < \omega$ such that any finite intersection of groups from this family is already an intersection of n of them.*

Proof. Suppose not, then for every $n < \omega$ there are $c_0, \dots, c_{n-1} \in C$ and $g_0, \dots, g_{n-1} \in G$ (in some model) such that $\varphi(g_i, c_j)$ holds iff $i \neq j$. For $s \subseteq n$, let $g_s = \prod_{i \in s} g_i$ (the order does not matter), then $\varphi(g_s, c_j)$ iff $j \notin s$—this is a contradiction. $\square$

In stable theories (which we shall not define here), the Baldwin–Saxl lemma is even stronger: every intersection of such a family is really a finite one (see [7, Proposition 1.4]).

The focus of this note is type definable groups in dependent theories, where such a proof does not work.

Definition 1.4. A *type definable group* for a theory T is a type—a collection $\Sigma(x)$ of formulas (maybe over parameters), and a formula $\nu(x, y, z)$, such that in the monster model $\mathfrak{C}$ of T, $\langle \Sigma(\mathfrak{C}), \nu \rangle$ is a group with ν defining the group operation (without loss of generality, $T \models \forall xy \exists^{\leqslant 1} z(\nu(x, y, z))$). We shall denote this operation by $\cdot$.

In stable theories, their analysis becomes easier as each type definable group is an intersection of definable ones (see [7]).

Remark 1.5. In this note we assume that G is a finitary type definable group, i.e. x above is a finite tuple.

Definition 1.6. Suppose $G \geqslant H$ are two type definable groups (H is a subgroup of G). We say that the index $[G : H]$ is *unbounded*, or ∞, if for any cardinality κ, there exists a model $M \models T$, such that $[G^M : H^M] \geqslant \kappa$. Equivalently (by the Erdős–Rado coloring theorem), this means that there exists (in $\mathfrak{C}$) a sequence of indiscernibles $\langle a_i \mid i < \omega \rangle$ (over the parameters defining G and H) such that $a_i \in G$ for all i, and $i < j \Rightarrow a_i \cdot a_j^{-1} \notin H$. In $\mathfrak{C}$, this means that $[G^{\mathfrak{C}} : H^{\mathfrak{C}}] = |\mathfrak{C}|$. When G and H are definable, then by compactness this is equivalent to the index $[G : H]$ being infinite.

So $[G : H]$ is *bounded* if it is not unbounded.

This leads to the following definition: