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Chain conditions in dependent groups

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ABSTRACT

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1. Introduction

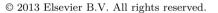
This note is about chain conditions in dependent, strongly dependent and strongly² dependent theories.

Throughout, all formulas will be first order, T will denote a complete first order theory, and \mathfrak{C} will be the monster model of T—a very big saturated model that contains all small models. We do not differentiate between finite tuples and singletons unless we state it explicitly.

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In this note we prove and disprove some chain conditions in type definable

and definable groups in dependent, strongly dependent and strongly² dependent



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Definition 1.1. A formula $\varphi(x, y)$ has the independence property in some model if for every $n < \omega$ there are $\langle a_i, b_s \mid i < n, s \subseteq n \rangle$ such that $\varphi(a_i, b_s)$ holds iff $i \in s$.

A (first order) theory T is dependent (sometimes also NIP) if it does not have the independence property: there is no formula $\varphi(x, y)$ that has the independence property in any model of T. A model M is dependent if Th(M) is.

A good introduction to dependent theories appears in [2], but we shall give an exact reference to any fact we use, so no prior knowledge is assumed.

What do we mean by a chain condition? Rather than giving an exact definition, we give an example of such a condition—the first one. It is the Baldwin–Saxl lemma, which we shall present with the (very easy and short) proof.

Definition 1.2. Suppose $\varphi(x, y)$ is a formula. Then if G is a definable group in some model, and for all $c \in C$, $\varphi(x, c)$ defines a subgroup, then $\{\varphi(\mathfrak{C}, c) \mid c \in C\}$ is a family of uniformly definable subgroups.

Lemma 1.3. (See [3].) Let G be a group definable in a dependent theory. Suppose $\varphi(x, y)$ is a formula and that $\{\varphi(x, c) \mid c \in C\}$ defines a family of subgroups of G. Then there is a number $n < \omega$ such that any finite intersection of groups from this family is already an intersection of n of them.

Proof. Suppose not, then for every $n < \omega$ there are $c_0, \ldots, c_{n-1} \in C$ and $g_0, \ldots, g_{n-1} \in G$ (in some model) such that $\varphi(g_i, c_j)$ holds iff $i \neq j$. For $s \subseteq n$, let $g_s = \prod_{i \in s} g_i$ (the order does not matter), then $\varphi(g_s, c_j)$ iff $j \notin s$ —this is a contradiction. \Box

In stable theories (which we shall not define here), the Baldwin–Saxl lemma is even stronger: every intersection of such a family is really a finite one (see [7, Proposition 1.4]).

The focus of this note is type definable groups in dependent theories, where such a proof does not work.

Definition 1.4. A type definable group for a theory T is a type—a collection $\Sigma(x)$ of formulas (maybe over parameters), and a formula $\nu(x, y, z)$, such that in the monster model \mathfrak{C} of T, $\langle \Sigma(\mathfrak{C}), \nu \rangle$ is a group with ν defining the group operation (without loss of generality, $T \models \forall xy \exists \leq 1 z(\nu(x, y, z))$). We shall denote this operation by \cdot .

In stable theories, their analysis becomes easier as each type definable group is an intersection of definable ones (see [7]).

Remark 1.5. In this note we assume that G is a finitary type definable group, i.e. x above is a finite tuple.

Definition 1.6. Suppose $G \ge H$ are two type definable groups (H is a subgroup of G). We say that the index [G : H] is *unbounded*, or ∞ , if for any cardinality κ , there exists a model $M \models T$, such that $[G^M : H^M] \ge \kappa$. Equivalently (by the Erdős–Rado coloring theorem), this means that there exists (in \mathfrak{C}) a sequence of indiscernibles $\langle a_i \mid i < \omega \rangle$ (over the parameters defining G and H) such that $a_i \in G$ for all i, and $i < j \Rightarrow a_i \cdot a_j^{-1} \notin H$. In \mathfrak{C} , this means that $[G^{\mathfrak{C}} : H^{\mathfrak{C}}] = |\mathfrak{C}|$. When G and H are definable, then by compactness this is equivalent to the index [G : H] being infinite.

So [G:H] is bounded if it is not unbounded.

This leads to the following definition:

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