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Large cardinals and basic sequences

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ABSTRACT

The purpose of this paper is to present several applications of combinatorial principles, well-known in Set Theory, to the geometry of infinite dimensional Banach spaces, particularly to the existence of certain basic sequences. We mention also some open problems where set-theoretical techniques are relevant.

1. Introduction

From the beginning of Banach space theory it was well understood how useful was to represent nicely the vectors of a given Banach space as sequences of scalars. This is for example the case when the space has a Schauder basis $(x_n)_n$: Every vector is represented as a unique (possibly) infinite linear combination of the vectors $(x_n)_{n\in\omega}$. However, not every Banach space, even the separable ones, has a Schauder basis. But fortunately, there are many of them. Indeed every infinite dimensional subspace contains itself a subspace with a Schauder basis, or every non-trivial weakly-null sequence has a Schauder basic subsequence. As it is expected, the more properties the basis has, the better understood the space may be. One of these properties is to be equivalent to the unit basis of the classical sequence spaces ℓ_p, $p \geqslant 1$, or c_0; another is the unconditionality. It was a central problem in the field to know if every infinite dimensional Banach space has an infinite dimensional subspace isomorphic to one of these classical sequence spaces. This was solved in the 70's by Tsirelson, who defined a space (interestingly, inspired by the method of forcing) not having subsymmetric sequences, and consequently not having isomorphic copies of the classical sequence spaces. It took a little more time until Gowers and Maurey found in the 90's a space without subspaces with unconditional bases. In the other direction, Ketonen showed in the 70's the relationship between the

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existence of unconditional basic sequences and large cardinals by proving that every Banach space whose density is an ω-Erdős cardinal always has a subspace with an unconditional basis. The key combinatorial feature used by Ketonen is the Ramsey property of the ω-Erdős cardinals. In this paper we explain this and some other applications of Ramsey properties of a cardinal κ, like the Polarized property $\mathrm{Pl}_2(\kappa)$, or the Free Set Property $\mathrm{Fr}(\kappa, \omega)$, and we see how they force a Banach space of density κ to have a subspace with an unconditional basis. We will also see how Ketonen's result can be improved to obtain that it is consistent relative to the existence of large cardinals that every Banach space of density at least ω_ω has a subspace with an unconditional basis. On the other hand, we give details of the constructions, using anti-Ramsey principles, of large sequences without unconditional basic subsequences. At the end of the paper we will also mention problems concerning the existence of certain uncountable sequence, and we will present a general approach to define generic spaces of density ω_1 lacking those uncountable sequences.

2. Basic notions and facts

Recall that a normed space $(X, \|\cdot\|)$ is a vector space X (over the real numbers $\mathbb{R}$ here) and a norm $\|\cdot\| : X \to \mathbb{R}$ on it, i.e.

(N.1) $\|\lambda x\| = |\lambda|\|x\|$ for every $x \in X$ and $\lambda \in \mathbb{R}$.
(N.2) $\|x + y\| \leqslant \|x\| + \|y\|$ for every $x, y \in X$.
(N.3) $\|x\| = 0$ iff $x = 0$.

The normed space $(X, \|\cdot\|)$ is a Banach space when the norm $\|\cdot\|$ is *complete*, i.e. Cauchy sequences are convergent. Well-know examples are $\mathbb{R}^n$ with the Euclidean norm $\|(a_i)_{i<n}\|_2 = (\sum_{i<n} |a_i|^2)^{\frac{1}{2}}$; the infinite dimensional separable *Hilbert space* $\ell_2 = \{(a_i)_{i\in\mathbb{N}}\colon (\sum_{i\in\mathbb{N}} |a_i|^2)^{\frac{1}{2}} < \infty\}$, with the Euclidean norm $\|(a_i)_{i\in\mathbb{N}}\|_2 = (\sum_{i\in\mathbb{N}} |a_i|^2)^{\frac{1}{2}}$; the ℓ_p spaces, for $p \geqslant 1$, $\ell_p = \{(a_i)_{i\in\mathbb{N}}\colon (\sum_{i\in\mathbb{N}} |a_i|^p)^{\frac{1}{p}} < \infty\}$, with the p-norm $\|(a_i)_{i\in\mathbb{N}}\|_p = (\sum_{i\in\mathbb{N}} |a_i|^p)^{\frac{1}{p}}$; the space $c_0 = \{(a_i)_{i\in\mathbb{N}}\colon \lim_{i\to\infty} a_i = 0\}$, with the sup-norm $\|(a_i)_{i\in\mathbb{N}}\|_\infty = \sup\{|a_i|\colon i \in \mathbb{N}\}$; the non-separable space $\ell_\infty = \{(a_i)_{i\in\mathbb{N}}\colon \sup_{i\in\mathbb{N}} |a_i| < \infty\}$, with the sup-norm; or for a compact space K, the space $C(K)$ of continuous functions on K, endowed with the sup-norm, $\|f\| = \sup\{|f(x)|\colon x \in K\}$ in particular, $C([0,1])$.

Some basic notions that we are going to use along this paper are the following: A Banach space is *infinite dimensional* if it is not a finite dimensional vector space. The *density* of a space X is its topological weight, i.e. the smallest cardinality of a dense subset of X. A *subspace* Y of X will be understood as a linear subspace of X, which is in addition *closed*. In particular Y with the norm $\|\cdot\|$ is also a Banach space. Given a subspace Y of X, the *quotient space* X/Y is the Banach space over the linear quotient, endowed with the norm $\|x + Y\| := d(x, Y)$. An *operator* $T : X \to Y$ between two spaces X and Y is a linear mapping which is continuous, or equivalently *bounded*, i.e., such that

$$\|T\| := \sup\bigl\{\|Tx\|\colon x \in X,\ \|x\| \leqslant 1\bigr\} < \infty.$$

An *isomorphic embedding* $T : X \to Y$ is a 1–1 operator such that $T(X)$ is a closed subspace of Y and the inverse $U : T(X) \to X$ is bounded. The dual X^* of a Banach space X is the space of all operators $f : X \to \mathbb{R}$. This is a Banach space with the norm

$$\|f\| := \sup\bigl\{\bigl\|f(x)\bigr\|\colon \|x\| \leqslant 1\bigr\}.$$

The elements of X^* are usually called *functionals*. The *weak topology* in X is the topology for which the basic open neighborhoods of a point $x \in X$ are $\{y \in X\colon \max_{i\leqslant n} |f_i(x) - f_i(y)| \leqslant \varepsilon\}$ for $f_0, \dots, f_n \in X^*$ and $\varepsilon > 0$. Similarly, the weak* topology in X^* is the topology with basic open neighborhoods of a functional f, the