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The descriptive set-theoretical complexity of the embeddability relation on models of large size

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ABSTRACT

We show that if κ is a weakly compact cardinal then the embeddability relation on (generalized) trees of size κ is invariantly universal. This means that for every analytic quasi-order R on the generalized Cantor space ${}^{\kappa}2$ there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ such that the embeddability relation on its models of size κ, which are all trees, is Borel bi-reducible (and, in fact, classwise Borel isomorphic) to R. In particular, this implies that the relation of embeddability on trees of size κ is complete for analytic quasi-orders on ${}^{\kappa}2$. These facts generalize analogous results for $\kappa = \omega$ obtained in Louveau and Rosendal (2005) [17] and Friedman and Motto Ros (2011) [6], and it also partially extends a result from Baumgartner (1976) [3] concerning the structure of the embeddability relation on linear orders of size κ.

1. Introduction

The aim of this paper is to establish a connection between descriptive set theory and (basic) model theory of uncountable models. In particular, we want to analyze the complexity of the embeddability relation on various classes of structures using typical methods of descriptive set theory, namely definable reducibility between quasi-orders and equivalence relations.

The embeddability relation, denoted in this paper by $\sqsubseteq$, is an important notion in model theory, but has also been widely considered in set theory. For example, in a long series of papers (see e.g. [21,18,15,5,25] and the references contained therein), it was determined for various cardinals κ whether there is a *universal* graph of size κ (i.e. a graph such that all other graphs of size κ embed into it) and, in the negative case,

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the possible size of a minimal *universal family*, i.e. of a family $\mathcal{D}$ of graphs of size κ with the property that for every other graph G of size κ there is $H \in \mathcal{D}$ such that $G \sqsubseteq H$. Another interesting example is contained in the paper [3], where Baumgartner shows that the embeddability relation on linear orders of size a regular cardinal κ is extremely rich and complicated (see Remark 9.6), a fact that should be contrasted with the celebrated Laver's proof [16] of the Fraïssé conjecture, which states that the embeddability relation on countable linear orders is a wqo.

Fix an infinite cardinal κ. Starting from the mentioned result from [3], in this work we will compare the complexity of the embeddability relation on various *elementary classes* of models, i.e. on the classes $\mathrm{Mod}^\kappa_\varphi$ of models of size κ of various $\mathcal{L}_{\kappa^+\kappa}$-sentences φ. A standard way to achieve this goal is to say that the embeddability relation $\sqsubseteq\restriction \mathrm{Mod}^\kappa_\varphi$ is no more complicated than the relation $\sqsubseteq\restriction \mathrm{Mod}^\kappa_\psi$ (where φ and ψ are two $\mathcal{L}_{\kappa^+\kappa}$-sentences) exactly when there is a "simply definable" reduction between $\sqsubseteq\restriction \mathrm{Mod}^\kappa_\varphi$ and $\sqsubseteq\restriction \mathrm{Mod}^\kappa_\psi$. This idea is precisely formalized in Definition 6.3 with the notion of *Borel reducibility* $\leqslant_B$ (and of the induced equivalence relation of *Borel bi-reducibility* $\sim_B$) between analytic quasi-orders.[2] This notion of reducibility was first introduced in [7] and [10] for the case $\kappa = \omega$. Our generalization to uncountable cardinals κ was independently introduced also in [8], where (among many other results) the complexity in terms of Shelah's stability theory of two first order theories T, T' is related to the relative complexity under $\leqslant_B$ of the corresponding isomorphism relations $\cong\restriction \mathrm{Mod}^\kappa_T$ and $\cong\restriction \mathrm{Mod}^\kappa_{T'}$ (for suitable uncountable cardinals κ).

The main result of this paper is the following.

Theorem 1.1. *Let κ be a weakly compact cardinal.*[3] *The embeddability relation on* (*generalized*) *trees of size κ is* (strongly) invariantly universal,[4] *i.e. for every analytic quasi-order R on a standard Borel κ-space*[5] *there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ all of whose models are trees such that R is Borel bi-reducible with* (*and, in fact, even classwise Borel isomorphic*[6] *to*) *the embeddability relation $\sqsubseteq\restriction \mathrm{Mod}^\kappa_\varphi$.*

Notice that since every relation of the form $\sqsubseteq\restriction \mathrm{Mod}^\kappa_\varphi$ is an analytic quasi-order, Theorem 1.1 actually yields a characterization of the class of analytic quasi-orders.

Corollary 1.2. *Let κ be a weakly compact cardinal. A binary relation R on a standard Borel κ-space is an analytic quasi-order if and only if there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ such that $R \sim_B \sqsubseteq\restriction \mathrm{Mod}^\kappa_\varphi$.*

Moreover, Theorem 1.1 obviously yields an analogous result for analytic equivalence relations, namely that the bi-embeddability relation on trees of size κ is (strongly) invariantly universal for the class of analytic equivalence relations on standard Borel κ-spaces.

Theorem 1.1 can be naïvely interpreted as saying that the embeddability relations (on elementary classes) are ubiquitous in the realm of analytic quasi-orders, and that given any "complexity" for an analytic quasi-order there is always an elementary class $\mathrm{Mod}^\kappa_\varphi$ such that $\sqsubseteq\restriction \mathrm{Mod}^\kappa_\varphi$ has exactly that complexity. So, in particular, there are elementary classes such that the corresponding embeddability relation is very simple (e.g. a linear order, a nonlinear bqo, an equivalence relation with any permitted[7] number of classes, and so on), and other elementary classes giving rise to a very complicated embeddability relation.

Moreover, since Theorem 1.1 establishes an exact correspondence between the structure of the embeddability relations (on elementary classes) under $\leqslant_B$ and the structure of analytic quasi-orders under $\leqslant_B$, any result concerning one of these two structures can be automatically transferred to the other one. For

[2] See Definition 6.1.
[3] See Definition 5.1.
[4] See Definitions 6.5 and 6.7.
[5] See Definition 3.6.
[6] See Definition 6.6.
[7] See e.g. [23].