Annals of Pure and Applied Logic 164 (2013) 1493–1506

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Goodstein sequences for prominent ordinals up to the ordinal of $\Pi^1_1-\mathrm{CA}_0$

Andreas Weiermann [a,1], Gunnar Wilken [b,*]

[a] *Department of Mathematics, Ghent University, Building S22, Krijgslaan 281, B 9000 Gent, Belgium*
[b] *Structural Cellular Biology Unit, Okinawa Institute of Science and Technology, 1919-1 Tancha, Onna-son, 904-0495 Okinawa, Japan*

ARTICLE INFO

Article history:
Received 14 February 2012
Received in revised form 10 May 2013
Accepted 16 May 2013
Available online 2 July 2013

MSC:
03F15
03D20
03E35
68Q42

Keywords:
Goodstein sequence
Proof-theoretic ordinal
Unprovability

ABSTRACT

We introduce strong Goodstein principles which are true but unprovable in strong impredicative theories like ID_n.

1. Introduction

Goodstein sequences provide examples for strictly mathematical statements which are true (by Goodstein, see [8]) but (according to Kirby and Paris, see [9]) not provable in PA. In the 80s several attempts have been made to define Goodstein principles capturing larger complexities using Π^1_2-logic. Unfortunately, even slight extensions of the original Goodstein principle led in some articles (see for instance [1]) to somewhat messy expositions which were not completely transparent, at least from our point of view.

Quite recently an alternative and transparent method to generate Goodstein principles has been provided by De Smet and Weiermann in [6]. Their Goodstein principles ranged in strength between Peano Arithmetic (PA) and the theory ID_1 of non-iterated monotone inductive definitions, and they asked whether an extension

* Corresponding author.
E-mail addresses: weierman@cage.ugent.be (A. Weiermann), wilken@oist.jp (G. Wilken).

[1] This author's research is supported in part by Fonds Wetenschappelijk Onderzoek (FWO) and the John Templeton Foundation. Parts of the research related to this article have been carried out during a visit of this author at the Isaac Newton Institute, Cambridge, UK in January 2012.

0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2013.06.019

to the theories ID_n was possible. In this article we provide an affirmative answer by elementary calculations based on Buchholz style tree ordinals and a trick suggested by Cichon, see [5].

There is some indication that Goodstein principles have no canonical extension to a strength beyond ID_ν and we expect having reached a canonical limit for strong Goodstein principles.

2. Tree ordinals

We introduce tree ordinals, following lecture notes by Wilfried Buchholz. Minor technical modifications are motivated by our specific purposes.

Definition 2.1. Inductive definition of classes $\mathbb{T}_i$, $i < \omega$, of tree ordinals.

1. $\mathbf{0} := () \in \mathbb{T}_i$.
2. $\boldsymbol{\alpha} \in \mathbb{T}_i \Rightarrow \boldsymbol{\alpha} + \mathbf{1} := (\boldsymbol{\alpha}) \in \mathbb{T}_i$.
3. $\forall n \in \mathbb{N}(\boldsymbol{\alpha}_n \in \mathbb{T}_i) \Rightarrow (\boldsymbol{\alpha}_n)_{n\in\mathbb{N}} \in \mathbb{T}_i$.
4. $j < i$ & $\forall \boldsymbol{\xi} \in \mathbb{T}_j(\boldsymbol{\alpha}_{\boldsymbol{\xi}} \in \mathbb{T}_i) \Rightarrow (\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\mathbb{T}_j} \in \mathbb{T}_i$.

The set of tree ordinals, denoted by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, etc., is thus given by

$$\mathbb{T}_{<\omega} := \bigcup_{i<\omega} \mathbb{T}_i.$$

We also use the notation $\mathbf{1} := (()) = \mathbf{0} + \mathbf{1}$.

Note that every $\boldsymbol{\alpha} \in \mathbb{T}_i$ is of a form $(\boldsymbol{\alpha}_\iota)_{\iota\in I}$ where I is one of the sets $\emptyset$, $\{0\}$, $\mathbb{N}$, or $\mathbb{T}_j$ for some $j < i$. We define

$$\left\|(\boldsymbol{\alpha}_\iota)_{\iota\in I}\right\| := \sup_{\iota\in I}\left(\|\boldsymbol{\alpha}_\iota\| + 1\right).$$

By transfinite induction on $\|\boldsymbol{\alpha}\|$ it is easy to show that $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_\iota)_{\iota\in I} \in \mathbb{T}_i$ implies $\boldsymbol{\alpha}_\iota \in \mathbb{T}_i$ for all $\iota \in I$.

We introduce the following abbreviations:

$$\underline{0} := \mathbf{0}, \qquad \underline{n+1} := \underline{n} + \mathbf{1}$$

and

$$\Omega_0 := (\underline{n})_{n\in\mathbb{N}}, \qquad \Omega_{i+1} := (\boldsymbol{\xi})_{\boldsymbol{\xi}\in\mathbb{T}_i},$$

so that $\Omega_i \in \mathbb{T}_i - \bigcup_{j<i} \mathbb{T}_j$. We will sometimes write ω for both $\underline{\omega} := \Omega_0$ and $\mathbb{N}$, assuming that ambiguity is excluded by context. Likewise, we will sometimes identify Ω_{i+1} with $\mathbb{T}_i$.

Addition is defined by

$$\boldsymbol{\alpha} + \mathbf{0} := \boldsymbol{\alpha}, \qquad \boldsymbol{\alpha} + (\boldsymbol{\beta}_\iota)_{\iota\in I} := (\boldsymbol{\alpha} + \boldsymbol{\beta}_\iota)_{\iota\in I} \quad \text{if } I \neq \emptyset,$$

consistent with the above definition of the special case $\boldsymbol{\alpha} + \mathbf{1}$, and multiples are defined by

$$\boldsymbol{\alpha} \cdot 0 := \mathbf{0}, \qquad \boldsymbol{\alpha} \cdot (n+1) := (\boldsymbol{\alpha} \cdot n) + \boldsymbol{\alpha}.$$

Proposition 2.2. *Let* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{T}_{<\omega}$.

1. $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{T}_i \Rightarrow \boldsymbol{\alpha} + \boldsymbol{\beta} \in \mathbb{T}_i$.
2. $\boldsymbol{\alpha} + (\boldsymbol{\beta} + \boldsymbol{\gamma}) = (\boldsymbol{\alpha} + \boldsymbol{\beta}) + \boldsymbol{\gamma}$.