Annals of Pure and Applied Logic 166 (2015) 261–273

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Topological dynamics for groups definable in real closed field

Ningyuan Yao *, Dongyang Long

School of Information Science and Technology, Sun Yat-sen University, China

ARTICLE INFO

Article history:
Received 21 July 2014
Received in revised form 13 October 2014
Accepted 14 October 2014
Available online 1 December 2014

MSC:
03C64

Keywords:
o-Minimality
Topological dynamics
Compact-torsion-free decomposition
Minimal flows
Ellis group

ABSTRACT

We study the definable topological dynamics of groups definable in an o-minimal expansion of an arbitrary real closed field M. For a definable group G which admits a compact-torsion-free decomposition $G = HK$, we give a description of the minimal subflow and Ellis group of the universal definable $G(M)$-flow $S_{G,ext}(M)$. This Ellis group is isomorphic to $N_G(H) \cap K(\mathbb{R})$, which extends the result of G. Jagiella from [7]. We also consider $SL(2, M)$ as an example, explaining the difference between the universal definable $SL(2, \mathbb{R})$-flow, $S_G(\mathbb{R})$ and the universal definable $G(M)$-flow, $S_{G,ext}(M)$ for an arbitrary model $M \succ \mathbb{R}$.

1. Introduction and preliminaries

The model theoretic approach to topological dynamics was introduced by Newelski in [10], where he studied the action a definable group $G(M)$ on its type space $S_G(M)$ and proved that $S_{G,ext}(M)$ is the universal definable $G(M)$-flow and hence an Ellis semigroup. Also in [10], Newelski conjectured that in the case of a theory with NIP, the Ellis groups of $S_{G,ext}(M)$ are isomorphic to G/G^{00}. The first counterexample of this conjecture is found in [5], where the authors showed that Newelski's conjecture fails in the case of $G = SL(2)$ and $M = \mathbb{R}$. Moreover, in [7] G. Jagiella provided a range of counterexamples by extending results in [5]. But in both papers, one works over the field of reals $\mathbb{R}$, where every externally definable subset of $\mathbb{R}$ is also a definable subset of $\mathbb{R}$. So $S_{G,ext}(\mathbb{R})$ equals to $S_G(\mathbb{R})$. This paper is inspired by the ideas of [5] as well as [7], and provides a broader range of such counterexamples by extending results in [7] to the context of an arbitrary real closed field M. Now we highlight our main result as follows:

* Corresponding author.
E-mail addresses: ningyuan.yao@gmail.com (N. Yao), issldy@mail.sysu.edu.cn (D. Long).

http://dx.doi.org/10.1016/j.apal.2014.10.004
0168-0072/

Theorem. *Let G be a group definable over $\mathbb{R}$ admitting a compact-torsion-free decomposition $G = HK$ with H torsion free and K definably compact, and M be an arbitrary elementary extension of $\mathbb{R}$. Then the Ellis group of $S_G(M^{ext})$ is algebraically isomorphic to $N_G(H) \cap K(\mathbb{R})$.*

In the rest of this introduction we give a description of the key aspects of the model-theoretic and topological dynamics context, as well as their interaction. References are [7,10,11].

In Section 2, we will turn to a class of definable group which admits so-called "compact-torsion-free decomposition", where we give precise definitions and prove some basic but nontrivial results.

In Section 3, we describe the minimal subflow and calculate the Ellis group of $S_G(M^{ext})$ (the notation will be defined later) for G admitting the compact-torsion-free decomposition and M an arbitrary elementary extension of $\mathbb{R}$. We find that the structure of minimal subflow is not preserved by different models. But the Ellis group is independent of the models.

In Section 4, we present the example of $G = SL(2)$, which illustrates the difference between $S_G(\mathbb{R})$ and $S_G(M^{ext})$ of an arbitrary model $M \succ \mathbb{R}$.

1.1. G-flows and its enveloping semigroup

Assume G is a group. By a (point-transitive) *G-flow* we mean a compact Hausdorff space X together with a left action of G on X by homeomorphism that contains a dense orbit. By a *subflow* of X we mean a closed subspace of X which is closed under the action of G. Since X is compact, there exist minimal subflows of X. The minimal subflows of X are very important objects, the one which are "dynamically indecomposable" and considered to be the most fundamental G-flows.

Let X^X denote the collection of all maps from X to itself, provided with the product topology. Then X^X is also a compact Hausdorff space by Tychonoff's theorem. Now each $g \in G$ defines a homeomorphism $\pi_g : X \mapsto X$, which is an element in X^X. Let $E(X)$ denote the closure of the set $\{\pi_g : g \in G\}$ in X^X. Then $E(X)$ together with the operation $*$ of function composition defines a semigroup structure $(E(X), *)$. We call $E(X)$ the *Ellis semigroup* of the G-flow X. For every $x \in X$, the closure of its G-orbit is exactly $E(X)(x) = \{f(x) : f \in E(X)\}$. $E(X)$ itself is a G-flow with function composition as the G-action. Every minimal subflow of the G-flow $E(X)$ is a minimal left ideal of the semigroup $E(X)$, and homeomorphic to each other as G-flows, i.e., for any minimal subflows I_1 and I_2 of $E(X)$, there exists a homeomorphism $f : I_1 \to I_2$ such that $f(gx) = gf(x)$ for all $x \in I_1$. We sometimes use the phrase *"minimal subflow of $E(X)$"* to denote the homeomorphism class of minimal subflows of $E(X)$. Every minimal subflow I is the closure of the G-orbit of every $p \in I$, hence is $E(X) * p$. By an *idempotent* of I we mean some $u \in I$ such that $u * u = u$. We denote the collection of all idempotents of I by $J(I)$. For any $u \in J(I)$, we have that $(u * I, *)$ is a group with u as its identity. All those groups are isomorphic to each other, even for different minimal left ideals. We call these groups the *ideal groups* and call their isomorphism class the *Ellis group* of $E(X)$. For more details, readers need to see Refs. [1,3].

1.2. The universal definable $G(M)$-flow $S_{G,ext}(M)$

Now we consider the topological dynamics in the model-theoretic context. We will assume a basic knowledge of model theory. Good references are [13] and [15]. We sometimes work in a sufficiently saturated model $\bar{M}$ of a theory T, in which every type over a small model M is realized. Definability usually means with parameters, and by a formula (or a partial type) we mean one definable with parameters from A, for A a subset of $\bar{M}$. By x, y, z we mean arbitrary n-variables and $a, b, c \in \bar{M}$ denote n-tuples in $\bar{M}^n$ with $n \in \mathbb{N}$.