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Definable closure in randomizations

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ABSTRACT

The randomization of a complete first order theory T is the complete continuous theory T^R with two sorts, a sort for random elements of models of T, and a sort for events in an underlying probability space. We give necessary and sufficient conditions for an element to be definable over a set of parameters in a model of T^R.

1. Introduction

A randomization of a first order structure $\mathcal{M}$, as introduced by Keisler [12] and formalized as a metric structure by Ben Yaacov and Keisler [4], is a continuous structure $\mathcal{N}$ with two sorts, a sort for random elements of $\mathcal{M}$, and a sort for events in an underlying atomless probability space. Given a complete first order theory T, the theory T^R of randomizations of models of T forms a complete theory in continuous logic, which is called the randomization of T. In a model $\mathcal{N}$ of T^R, for each n-tuple $\vec{a}$ of random elements and each first order formula $\varphi(\vec{v})$, the set of points in the underlying probability space where $\varphi(\vec{a})$ is true is an event denoted by $[\![\varphi(\vec{a})]\!]$.

In a first order structure $\mathcal{M}$, an element b is *definable over* a set A of elements of $\mathcal{M}$ (called parameters) if there is a tuple $\vec{a}$ in A and a formula $\varphi(u, \vec{a})$ such that

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$$\mathcal{M} \models (\forall u)\big(\varphi(u, \vec{a}) \leftrightarrow u = b\big).$$

In a general metric structure $\mathcal{N}$, an element b is said to be *definable over* a set of parameters A if there is a sequence of tuples $\vec{a}_n$ in A and continuous formulas $\Phi_n(x, \vec{a}_n)$ whose truth values converge uniformly to the distance from x to b. In this paper we give necessary and sufficient conditions for definability in a model of the randomization theory T^R. These conditions can be stated in terms of sequences of first order formulas.

In Theorem 3.1.2, we show that an event E is definable over a set A of parameters if and only if it is the limit of a sequence of events of the form $[\![\varphi_n(\vec{a}_n)]\!]$, where each φ_n is a first order formula and each $\vec{a}_n$ is a tuple from A.

In Theorem 3.3.6, we show that a random element b is definable over a set A of parameters if and only if b is the limit of a sequence of random elements b_n such that for each n,

$$[\![(\forall u)\big(\varphi_n(u, \vec{a}_n) \leftrightarrow u = b_n\big)]\!]$$

has probability one for some first order formula $\varphi_n(u, \vec{v})$ and a tuple $\vec{a}_n$ from A.

Our principal aim in this paper is to lay the groundwork for the study of independence relations in randomizations, that will appear in a forthcoming paper. However, in Section 4 of this paper we will give some more modest consequences of our results in the special case that the underlying first order theory T is $\aleph_0$-categorical.

Continuous model theory in its current form is developed in the papers [3] and [5]. The papers [8–10] deal with definability questions in metric structures. Randomizations of models are treated in [1,2,4,7,11], and [12].

2. Preliminaries

We refer to [3] and [5] for background in continuous model theory, and follow the notation of [4]. We assume familiarity with the basic notions about continuous model theory as developed in [3], including the notions of a theory, structure, pre-structure, model of a theory, elementary extension, isomorphism, and κ-saturated structure. In particular, the universe of a pre-structure is a pseudo-metric space, the universe of a structure is a complete metric space, and every pre-structure has a unique completion. In continuous logic, formulas have truth values in the unit interval $[0, 1]$ with 0 meaning true, the connectives are continuous functions from $[0, 1]^n$ into $[0, 1]$, and the quantifiers are sup and inf. A *tuple* is a finite sequence, and $A^{<\mathbb{N}}$ is the set of all tuples of elements of A.

2.1. The theory T^R

We assume throughout that L is a finite or countable first order signature, and that T is a complete theory for L whose models have at least two elements.

The *randomization signature* L^R is the two-sorted continuous signature with sorts $\mathbb{K}$ (for random elements) and $\mathbb{B}$ (for events), an n-ary function symbol $[\![\varphi(\cdot)]\!]$ of sort $\mathbb{K}^n \to \mathbb{B}$ for each first order formula φ of L with n free variables, a $[0, 1]$-valued unary predicate symbol μ of sort $\mathbb{B}$ for probability, and the Boolean operations $\top, \bot, \sqcap, \sqcup, \neg$ of sort $\mathbb{B}$. The signature L^R also has distance predicates $d_{\mathbb{B}}$ of sort $\mathbb{B}$ and $d_{\mathbb{K}}$ of sort $\mathbb{K}$. In L^R, we use $\mathsf{B}, \mathsf{C}, \ldots$ for variables or parameters of sort $\mathbb{B}$. $\mathsf{B} \doteq \mathsf{C}$ means $d_{\mathbb{B}}(\mathsf{B}, \mathsf{C}) = 0$, and $\mathsf{B} \sqsubseteq \mathsf{C}$ means $\mathsf{B} \doteq \mathsf{B} \sqcap \mathsf{C}$.

A pre-structure for T^R will be a pair $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ where $\mathcal{K}$ is the part of sort $\mathbb{K}$ and $\mathcal{B}$ is the part of sort $\mathbb{B}$. The *reduction* of $\mathcal{P}$ is the pre-structure $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$ obtained from $\mathcal{P}$ by identifying elements at distance zero in the metrics $d_{\mathbb{K}}$ and $d_{\mathbb{B}}$, and the associated mapping from $\mathcal{P}$ onto $\mathcal{N}$ is called the *reduction*