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Grothendieck rings of theories of modules

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ABSTRACT

The model-theoretic Grothendieck ring of a first order structure, as defined by Krajiček and Scanlon, captures some combinatorial properties of the definable subsets of finite powers of the structure. In this paper we compute the Grothendieck ring, $K_0(M_{\mathcal{R}})$, of a right $\mathcal{R}$-module M, where $\mathcal{R}$ is any unital ring. As a corollary we prove a conjecture of Prest that $K_0(M_{\mathcal{R}})$ is non-trivial, whenever M is non-zero. The main proof uses various techniques from simplicial homology and lattice theory to construct certain counting functions.

1. Introduction

1.1. Historical background

Krajiček and Scanlon introduced the concept of the model-theoretic Grothendieck ring of a structure in [13]. Amongst many other results, they proved that such a Grothendieck ring is nontrivial if and only if the definable subsets of the structure satisfy a version of the combinatorial pigeonhole principle, called the "onto pigeonhole principle". Grothendieck rings have been studied for various rings and fields considered as models of a first order theory (see [13,3–6]) and they are found to be trivial in many cases (see [3,4]).

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Prest conjectured that in stark contrast to the case of rings, for any ring $\mathcal{R}$, the Grothendieck ring of a nonzero right $\mathcal{R}$ module $M_{\mathcal{R}}$, denoted $K_0(M_{\mathcal{R}})$, is nontrivial. Perera [17] investigated the problem in his doctoral thesis but found only a partial solution. He showed that elementarily equivalent modules have isomorphic Grothendieck rings, which is not the case for general structures, and he showed that the Grothendieck ring for modules over semisimple rings are polynomial rings in finitely many variables over the ring of integers.

1.2. Contributions of the paper

In this paper we compute the Grothendieck ring for arbitrary modules over arbitrary rings and show that they are quotients of monoid rings $\mathbb{Z}[\mathcal{X}]$, where $\mathcal{X}$ is the multiplicative monoid of isomorphism classes of fundamental definable subsets of the module – the *pp*-definable subgroups. This is the content of the main theorem, Theorem 5.2.3, which also describes the 'invariants ideal' – the ideal of the monoid ring that codes indices of pairs of *pp*-definable subgroups. It should be noted that the proof gives an explicit description of the class $[D] \in K_0(M)$ of a definable set D. We further show (Corollary 5.2.11) that there is a split embedding $\mathbb{Z} \to K_0(M)$, whenever the module M is nonzero, proving Prest's conjecture.

The proof of the main theorem uses inputs from various mathematical areas such as model theory, algebra, combinatorics and algebraic topology. Careful analysis of the meet-semilattice of *pp*-sets using Euler characteristics of abstract simplicial complexes yields a family of counting measures on definable sets; the Grothendieck ring bundles up such measures.

A special case of the main theorem (Theorem 4.1.2) is proved at the end of Section 4. The special case assumes that the theory T of the module M satisfies the model theoretic condition $T = T^{\aleph_0}$. This condition is equivalent to the statement that the invariants ideal is trivial. The reader should note that the proof of the general case of the main theorem is not given in full detail since it develops along lines similar to the special case and uses only a few modifications which are indicated to incorporate the invariants ideal.

1.3. Canonical forms for definable sets

Holly [11] described a canonical form for definable subsets of algebraically closed valued fields by forming swiss cheeses – discs or balls with finitely many holes removed. She used the canonical form to prove the elimination of imaginaries in one dimension. Adler used the swiss cheeses to study definable sets in some classes of VC-minimal theories in [1]. Flenner and Guingona [9] extracted the notion of a directed family of sets from Adler's work and used it to obtain uniqueness results on representations of sets constructible in a directed family. A directed family of sets is a meet-semilattice in which every two elements with non-trivial intersections are comparable. They introduced the notion of packability as the dividing line between absolute uniqueness and optimal uniqueness of representations.

The fundamental theorem of the model theory of modules (Theorem 2.5.5) states that every definable set is a boolean combination of *pp*-definable sets, but such a boolean combination is far from being unique. Under certain extra conditions on the theory of the module, we achieve a 'uniqueness' result as a by-product of the theory we develop. We call this result the 'cell decomposition theorem' (Theorem 6.3.4) which states that definable sets can be represented uniquely using the meet-semilattice of *pp*-definable sets provided the theory T of the module satisfies $T = T^{\aleph_0}$. This model-theoretic condition is analogous to the notion of unpackability in [9, § 2] and Theorem 6.3.4 generalises [9, Corollary 2.3].

Though the cell decomposition theorem is not used directly in any other proof, its underlying idea is one of the most important ingredients of the main proof. Based on this idea, we define various classes of definable sets of increasing complexity, namely *pp*-sets, convex sets, blocks and cells. The terms ball and swiss cheese in [9] correspond to the terms *pp*-sets and blocks in our setting. Our strategy to prove every result about a general definable set is to prove it first for convex sets, then blocks and then cells. We deal