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Relativization makes contradictions harder for Resolution ☆

Stefan Dantchev [a], Barnaby Martin [b,*]

[a] School of Engineering and Computing Sciences, Durham University, Science Labs, South Road, Durham DH1 3LE, UK
[b] School of Science and Technology, Middlesex University, The Burroughs, Hendon, London NW4 4BT, UK

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ABSTRACT

We provide a number of simplified and improved separations between pairs of Resolution-with-bounded-conjunction refutation systems, $\mathrm{Res}(d)$, as well as their tree-like versions, $\mathrm{Res}^*(d)$. The contradictions we use are natural combinatorial principles: the *Least number principle*, LNP_n and an ordered variant thereof, the *Induction principle*, IP_n.
LNP_n is known to be easy for Resolution. We prove that its relativization is hard for Resolution, and more generally, the relativization of LNP_n iterated d times provides a separation between $\mathrm{Res}(d)$ and $\mathrm{Res}(d+1)$. We prove the same result for the iterated relativization of IP_n, where the tree-like variant $\mathrm{Res}^*(d)$ is considered instead of $\mathrm{Res}(d)$.
We go on to provide separations between the parameterized versions of $\mathrm{Res}(1)$ and $\mathrm{Res}(2)$. Here we are able again to use the relativization of the LNP_n, but the classical proof breaks down and we are forced to use an alternative. Finally, we separate the parameterized versions of $\mathrm{Res}^*(1)$ and $\mathrm{Res}^*(2)$. Here, the relativization of IP_n will not work as it is, and so we make a vectorizing amendment to it in order to address this shortcoming.

1. Introduction

We study the power of relativization in *Propositional proof complexity*, i.e. we are interested in the following question: given a propositional proof system is there a first-order (FO) sentence which is easy but whose relativization is hard (within the system)? The main motivation for studying relativization comes from a work of Krajíček, [19]. He defines a combinatorics of FO structure and a relation of covering between FO structures and propositional proof systems. The combinatorics contains all the sentences easy for the proof system. On the other hand, as defined in [19], it is closed under relativization. Thus the existence of a sentence, which is easy but whose relativization is hard, for the underlying proof system, would imply that it is impossible to capture the class of "easy" sentences by a combinatorics. Ideas of relativization have also

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* Corresponding author.

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appeared in [11,2]. The proof, in fact refutation, system we consider is Resolution-with-bounded-conjunction, denoted Res(d) and introduced by Krajíček in [18]. It is an extension of Resolution in which conjunctions of up to d literals are allowed instead of single literals. The tree-like version of Res(d) is usually denoted Res*(d). Krajíček proved that tree-like Resolution, and even Res*(d), have combinatorics associated with it. This follows also from Riis's complexity gap theorem for tree-like Resolution [23], and shows that the sentences, easy for tree-like Resolution, remain easy after having been relativized.

The next natural system to look at is Resolution. It is stronger than Res*(d) for any d, $1 \leqslant d \leqslant n$ (equivalent to Res*(n), in fact, where n is the number of variables), and yet weak enough so that one could expect that it can easily prove some property of the whole universe, but cannot prove it for an arbitrary subset. As we show in the paper, this is indeed the case. The example is very natural, the *Least number principle*, LNP$_n$. The contradiction LNP$_n$ asserts that a partial n-order has no minimal element. In the literature it enjoys a myriad of alternative names: the *Graph ordering principle* GOP, *Ordering principle* OP and *Minimal element principle* MEP. Where the order is total it is also known as TLNP and GT. It is not hard to see that LNP$_n$ is easy for Resolution [7], and we prove that its relativization RLNP$_n$ is hard. A more general result has been proven in [25]; however the lower bound there is weaker. We also consider iterated relativization, and show that the dth iteration d-RLNP$_n$ is hard for Res(d), but easy for Res($d+1$). We go on to consider the relativization question for Res*(d), where the FO language is enriched with a built-in order. The complexity gap theorem does not hold in this setting [11], though we are able to show that relativization again makes some sentences harder. A variant of the *Induction Principle* gives the contradiction IP$_n$, saying that there is a property which: holds for the minimal element; if it holds for a particular element, there is a bigger one for which the property holds, too; and the property does not hold for the maximal element. We prove that the dth iteration of the relativization of the Induction principle, d-RIP$_n$, is easy for Res*($d+1$), but hard for Res*(d). More precisely, our results are the following:

1. Any Resolution refutation of RLNP$_n$ is of size $2^{\Omega(n)}$. Firstly, this answers positively to Krajíček's question. Secondly, observing that RLNP$_n$ has an $\mathrm{O}(n^3)$-size refutation in Res(2), we get an exponential separation between Resolution and Res(2). A similar result was proved in [25] (see also [1] for a weaker, quasi-polynomial, separation). Our proof is quite simple compared with that of [25], where this separation is a corollary of a more general result, and our lower bound is stronger.
2. d-RLNP$_n$ has an $\mathrm{O}(dn^3)$-size refutation in Res($d+1$), but requires $2^{\Omega(n^{\epsilon})}$-size refutation in Res(d), where ϵ is a constant dependent on d. These separations were first proved in [25]. As a matter of fact, we use their method but our tautologies are more natural, and our proof is a little simpler.
3. d-RIP$_n$ has an $\mathrm{O}(dn^2)$-size Res*($d+1$) refutation, but requires Res*(d) refutations of size $2^{\Omega(\frac{n}{d})}$. This holds for any d, $0 \leqslant d \leqslant n$. A similar result was proven in [15]. Again, our tautologies are more natural, while the proof is simpler.

The second part of the paper is in the area of *Parameterized proof complexity*, a program initiated in [12], which generally aims to gain evidence that W[i] is different from FPT. Typically, i is so that the former is W[2], though—in the journal version [13] of [12]—this has been W[SAT] and—in the note [20]—W[1] was entertained. In the W[2] context, parameterized refutation systems aim at refuting *parameterized contradictions* which are pairs $(\mathcal{F}, k)$ in which $\mathcal{F}$ is a propositional CNF with no satisfying assignment of weight $\leqslant k$. Several parameterized (hereafter often abbreviated as "p-") proof systems are discussed in [12,3,6]. The lower bounds in [12], [3] and [6] amount to proving that the systems p-tree-Resolution, p-Resolution and p-bounded-depth Frege, respectively, are not *fpt-bounded*. Indeed, this is witnessed by the *Pigeonhole principle*, and so holds even when one considers parameterized contradictions $(\mathcal{F}, k)$ where $\mathcal{F}$ is itself an actual contradiction. Such parameterized contradictions are termed "*strong*" in [6], in which the authors suggest these might be the only parameterized contradictions worth considering, as general lower bounds—even in p-bounded-depth Frege—are trivial (see [6]). We sympathize with this outlook, but remind that there are