Annals of Pure and Applied Logic 165 (2014) 858–879

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Reverse mathematics and initial intervals ☆

Emanuele Frittaion, Alberto Marcone *

Dipartimento di Matematica e Informatica, Università di Udine, 33100 Udine, Italy

ARTICLE INFO

Article history:
Received 3 April 2013
Received in revised form 29 October 2013
Accepted 14 November 2013
Available online 6 December 2013

MSC:
primary 03B30
secondary 03F35, 06A07, 06A06

Keywords:
Reverse mathematics
Partial order
Initial interval
Scattered
Antichain

ABSTRACT

In this paper we study the reverse mathematics of two theorems by Bonnet about partial orders. These results concern the structure and cardinality of the collection of initial intervals. The first theorem states that a partial order has no infinite antichains if and only if its initial intervals are finite unions of ideals. The second one asserts that a countable partial order is scattered and does not contain infinite antichains if and only if it has countably many initial intervals. We show that the left to right directions of these theorems are equivalent to ACA_0 and ATR_0, respectively. On the other hand, the opposite directions are both provable in WKL_0, but not in RCA_0. We also prove the equivalence with ACA_0 of the following result of Erdös and Tarski: a partial order with no infinite strong antichains has no arbitrarily large finite strong antichains.

1. Introduction

In this paper we study from the viewpoint of reverse mathematics some theorems dealing with the structure and the cardinality of the collection of initial intervals (also called downward closed subsets) in a partial order. Recall that an ideal is an initial interval such that every pair of elements is compatible (i.e. has a common upper bound) in the interval.

The first result is a characterization of partial orders with no infinite antichains in terms of the decomposition of initial intervals into union of ideals. It is due to Bonnet [2, Lemma 2] and can be found in Fraïssé's monograph [6, §4.7.2]:

Theorem 1.1. *A partial order has no infinite antichains if and only if every initial interval is a finite union of ideals.*

☆ We thank Gregory Igusa, Matthew Hendtlass and Henry Towsner for useful discussions about the subject. In particular Igusa suggested the details of the proof of Lemma 5.11. We also thank the anonymous referees for a careful reading of the paper and several useful suggestions.

* Corresponding author.
E-mail addresses: emanuele.frittaion@uniud.it (E. Frittaion), alberto.marcone@uniud.it (A. Marcone).

http://dx.doi.org/10.1016/j.apal.2013.11.002

In [9] Theorem 1.1 is attributed to Erdös and Tarski because its 'hard' (left to right) direction can be deduced quite easily from the following result, which is part of [5, Theorem 1]:

Theorem 1.2. *If a partial order has no infinite strong antichains then it has no arbitrarily large finite strong antichains.*

Here, by strong antichain we mean a set of pairwise incompatible (and not only incomparable, as in antichain) elements. (Notice that Erdös and Tarski work with what we would call filters and final intervals.)

An intermediate step between Theorems 1.2 and 1.1 is the following characterization of partial orders with no infinite strong antichains:

Theorem 1.3. *A partial order has no infinite strong antichains if and only if it is a finite union of ideals.*

Our proof of Lemma 4.2 shows how to deduce the left to right direction of Theorem 1.3 from Theorem 1.2.

In [2] Theorem 1.1 is a step in the proof of the following result, which is also featured in Fraïssé's monograph [6, §6.7]:

Theorem 1.4. *If an infinite partial order P is scattered (i.e. there is no embedding of the rationals into P) and has no infinite antichains, then the set of initial intervals of P has the same cardinality of P.*

The converse of Theorem 1.4 is in general false, but it holds when $|P| < 2^{\aleph_0}$, and in particular when P is countable:

Theorem 1.5. *A countable partial order is scattered and has no infinite antichains if and only if it has countably many initial intervals.*

The program of reverse mathematics ([10] is the basic reference) gauges the strength of mathematical theorems by means of the subsystems of second order arithmetic necessary for their proofs. This approach allows only the study of statements about countable (or countably coded) objects. We therefore study the strength of Theorem 1.5 and of the restrictions of Theorems 1.1, 1.2 and 1.3 to countable partial orders. We notice that [5,2,6] put no restriction on the cardinality of the partial order and therefore often use set-theoretic techniques which are not available in (subsystems of) second order arithmetic. On the other hand we can always assume that the partial orders are defined on a subset of the set of the natural numbers, and this is on occasion helpful.

Since Theorems 1.1, 1.3, and 1.5 are equivalences, we study separately the two implications, which turn out to have different axiomatic strengths. In particular, the 'easy' (right to left) directions of Theorems 1.1 and 1.5 are quite interesting from the viewpoint of reverse mathematics and we are not able to settle the problem of establishing their strength, leaving open the possibility that they have strength intermediate between RCA_0 and WKL_0.

We assume familiarity with the 'big five' of reverse mathematics, namely, in order of increasing strength, RCA_0, WKL_0, ACA_0, ATR_0, and $\boldsymbol{\Pi}^1_1$-CA_0.

We now state our main results and at the same time describe the organization of the paper. In Section 2 we establish our notation and terminology and recall some basic results. In Section 3 we prove a couple of technical lemmas that are useful later on.

In Section 4 we consider Theorem 1.2 and the left to right directions of Theorems 1.1, 1.3, and 1.5. Section 4.1 culminates in Theorem 4.5 where we prove, over RCA_0, the equivalence of ACA_0 with each of the three statements:

- in a countable partial order with no infinite antichains every initial interval is a finite union of ideals;