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Omitting types for infinitary [0, 1]-valued logic ☆

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ABSTRACT

We describe an infinitary logic for metric structures which is analogous to $\mathcal{L}_{\omega_1,\omega}$. We show that this logic is capable of expressing several concepts from analysis that cannot be expressed in finitary continuous logic. Using topological methods, we prove an omitting types theorem for countable fragments of our infinitary logic. We use omitting types to prove a two-cardinal theorem, which yields a strengthening of a result of Ben Yaacov and Iovino concerning separable quotients of Banach spaces.

0. Introduction

In this paper we study an infinitary logic for structures based on metric spaces. Our logic is the natural generalization of $\mathcal{L}_{\omega_1,\omega}$ to the setting of metric structures, and our main result is an Omitting Types Theorem for this logic.

Real-valued logics have had a variety of applications in analysis, beginning with the introduction of ultrapowers of Banach spaces by Dacunha-Castelle and Krivine in [11]. Krivine, and later Stern, used this approach to solve important problems in functional analysis [28,30,38]. See also [26,27,29]. In [9] Chang and Keisler develop a general framework for continuous model theory with truth values in a fixed compact Hausdorff space K, with the case $K = [0,1]$ being the motivating example. In recent years there has been a considerable amount of activity in the [0, 1]-valued logic known as first-order continuous

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logic, which Ben Yaacov and Usvyatsov introduced in [4] as a reformulation of Henson's logic for Banach spaces (see [20]) in the framework of [9]. See [5] for a self-contained introduction to first-order continuous logic.

In this paper we extend the $[0,1]$-valued logics mentioned above by allowing formulas with conjunctions and disjunctions of countable length, which makes our logic a $[0,1]$-valued version of $\mathcal{L}_{\omega_1,\omega}$. Including infinitary formulas in our logic allows us to axiomatize important classes of structures from functional analysis (see Section 2). Our logic does not satisfy the compactness theorem, but as our main result shows, countable fragments of this logic do satisfy the classical Omitting Types Theorem.

In the $[0,1]$-valued setting we say that a type is *principal* over a theory T if there is a formula φ consistent with T, and an approximation φ' of φ, such that in models of T elements satisfying φ' are realizations of Σ (see Definition 4.10). Our main result is the following:

Theorem A. *Let S be a metric signature, and let L be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(S)$. Let T be an L-theory. For each $n < \omega$, let Σ_n be a type of T that is not principal over T. Then there is a model of T that omits every Σ_n.*

Each type Σ_n is in finitely many variables, but the number of variables may increase with n. For certain fragments of $\mathcal{L}_{\omega_1,\omega}$, which we call *continuous fragments*, Theorem A implies an Omitting Types Theorem in which the resulting model is based on a complete metric space (Proposition 4.13). This latter version generalizes Henson's Omitting Types Theorem to infinitary languages (see [3]). Henson's theorem is the motivation for recent uses of omitting types to characterize certain classes of operator algebras [8].

In the finitary setting it is straightforward to generalize the Omitting Types Theorem to uncountable languages (see [10, Theorem 2.2.19]). However, the main result of [7] shows that Theorem A cannot be generalized to arbitrary uncountable fragments of $\mathcal{L}_{\omega_1,\omega}$.

Our approach is topological, based on the connection between omitting types and the Baire Category Theorem. The Omitting Types Theorem for the classical (discrete) $\mathcal{L}_{\omega_1,\omega}$ is originally due to Keisler in [23]. Later Morley [32] obtained the same result by showing that a relevant topological space is metrizable by a complete metric. In our proof of Theorem A we avoid the issue of metrizability by instead working with a topological notion of completeness. Applying our proof in the classical setting thus gives a simplification of Morley's argument. We have learned from Iovino [21] that Caicedo has independently, in unpublished work, obtained the conclusion of Theorem A by adapting Morley's argument.

As applications of Theorem A we prove a $[0,1]$-valued version of Keisler's two-cardinal theorem (Theorem 5.3), and we extend a result of Ben Yaacov and Iovino from [2] regarding non-trivial separable quotients of Banach spaces (Corollary 5.4).

The paper requires only basic knowledge of classical first-order model theory. In Section 1 we provide the needed background material from both general topology and the model theory of metric structures. In Section 2 we introduce our $[0,1]$-valued version of $\mathcal{L}_{\omega_1,\omega}$ and state some of its basic properties. The topological spaces we will use in the proof of Theorem A are described in Section 3. In Section 4 we prove that the topological space from the previous section satisfy a topological completeness property which implies that they are Baire spaces. We then show that these topological results imply Theorem A. Section 5 contains the two-cardinal theorem and the applications to separable quotients of Banach spaces.

1. Preliminaries

1.1. Topological preliminaries

The spaces we will use are not T_0, but do have other separation properties. In particular, we say that a space is completely regular to mean that points and closed sets can be separated by continuous functions,