Annals of Pure and Applied Logic 166 (2015) 1–28

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Second-order propositional modal logic and monadic alternation hierarchies

Antti Kuusisto [a,b,*]

[a] *Institute of Computer Science, University of Wrocław, Poland*
[b] *School of Information Sciences, University of Tampere, Finland*

ARTICLE INFO

Article history:
Received 23 July 2013
Received in revised form 21 July 2014
Accepted 24 July 2014
Available online 2 September 2014

MSC:
03B45
03C80
03B15

Keywords:
Modal logic
Propositional quantifiers
Monadic second-order logic
Alternation hierarchies
Second-order logic

ABSTRACT

We establish that the quantifier alternation hierarchy of formulae of second-order propositional modal logic (SOPML) induces an infinite corresponding semantic hierarchy over the class of finite directed graphs. This solves an open problem of van Benthem (1985) [5] and ten Cate (2006) [11]. We also identify modal characterizations of the expressive power of second-order logic (SO) and monadic second-order logic (MSO) in terms of extensions of modal logic with second-order quantification.

1. Introduction

In this paper we investigate the expressive power of second-order propositional modal logic (SOPML), which is the system obtained by extending ordinary modal logic with propositional quantifiers ranging over sets of domain elements. Modal logics with propositional quantifiers have been investigated by a variety of researchers with different kinds of motivations, see [4,6,10,11,15,16,19,30,33] for example, and interestingly, also [20].

Johan van Benthem [5] and Balder ten Cate [11] raise the question whether the prenex quantifier alternation hierarchy of SOPML formulae induces an infinitely ascending corresponding hierarchy of definable classes of Kripke frames. This is an interesting question, especially as ten Cate shows in [11] that formulae of SOPML admit a prenex normal form representation. We show that the semantic counterpart of the

* Correspondence to: Institute of Computer Science, University of Wrocław, Poland.
E-mail address: antti.j.kuusisto@gmail.com.

http://dx.doi.org/10.1016/j.apal.2014.08.003
0168-0072/

syntactic quantifier alternation hierarchy of SOPML is infinite over the class of finite directed graphs. This automatically implies that the semantic hierarchy is infinite over the class of Kripke frames.

Second-order alternation hierarchies have received a significant amount of attention in finite model theory, see [28,29,26,31,32,34] for example. The most important result we will use in investigating quantifier alternation of SOPML is a theorem of Schweikardt from [31,32] stating that the alternation hierarchy of monadic second-order logic (MSO) is strict over the class of *grids*. Inspired by the approach of Matz and Thomas in [29], we employ a strategy loosely based on *strong first-order reductions* in order to transfer the result of Schweikardt from the context of grids to the context of a special class of finite directed graphs that we define. Over this class of finite directed graphs, the expressive power of SOPML coincides with that of MSO, and hence we easily obtain the desired result that the alternation hierarchy of SOPML is infinite over finite directed graphs. The precise definition of strong first-order reductions (found in [28]) is of no importance to the investigations below, as we give a self-contained and detailed exposition of all our results.

As a by-product of the investigations concerning alternation hierarchies, we obtain characterizations of monadic second-order logic and second-order logic in terms of extensions of ordinary modal logic with second-order quantifiers. We define a translation of MSO sentences to equivalent formulae of second-order propositional modal logic with the global modality (SOPMLE). The translation is a variant of a translation of ten Cate in [11]. The translation establishes that the expressive power of SOPMLE over finite/arbitrary relational structures coincides with that of MSO. A trivial adaptation of the related argument shows that replacing the global modality with the difference modality does not change the picture. We also define a translation of SO into *second-order modal logic* SOML. The logic SOML is the extension of *polyadic modal logic*[1] with quantification of both accessibility relations and proposition symbols.

Our modal characterizations of MSO and SO could turn out interesting from the point of view of finite model theory. For example, the modal characterization of MSO immediately suggest alternative approaches to the Ehrenfeucht–Fraïssé game for MSO (see [25] for the definition of the standard MSO game).

The paper is structured as follows. In Section 2 we fix the notational conventions and discuss a number of preliminary technical issues. In Section 3 we show that MSO = SOPMLE with regard to expressive power. Employing an approach analogous to that in Section 3, we then define in Section 4 a special class of directed graphs over which MSO and SOPML coincide in expressive power. In Section 5 we first work with MSO, transferring the result of Schweikardt to the context of the newly defined special class of directed graphs. Then, using the connection created in Section 4, we finally establish that the SOPML alternation hierarchy is infinite over directed graphs. In Section 6 we show that SO = SOML with regard to expressivity.

The current paper is the journal version of the conference article [21], which constitutes a part of the author's PhD thesis [22].

2. Preliminary definitions

In this section we introduce technical notions that occupy a central role in the investigations below.

We let $\mathbb{N}$ denote the set $\{0, 1, 2, ...\}$ of natural numbers. For $k \in \mathbb{N}$, we let $\mathbb{N}_{\geq k}$ denote the set of natural numbers greater or equal to k.

2.1. Syntax and semantics

With a model we mean a first-order model of predicate logic (see [14]), and we restrict attention to models associated with a vocabulary containing only relation symbols and possibly also constant symbols.

[1] Polyadic modal logics are modal logics with accessibility relations of arities higher than two. See Section 2.1 for the related definitions.