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Definable normal measures [☆]

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ABSTRACT

A normal measure U on a measurable cardinal κ cannot be definable over $H(\kappa^+)$, as otherwise it would belong to its own ultrapower. In this article we show that it may however be Δ_1 definable over $H(\kappa^{++})$ when the GCH fails at κ. In [7] it is shown that there can be a unique normal measure on κ when the GCH fails at κ; we show here that this unique normal measure can in addition be Δ_1 definable over $H(\kappa^{++})$.

1. Introduction and preliminaries

Definability is a central theme in set theory and in this paper we examine it in the large cardinal setting. Important work in inner model theory (see [10]) reveals that the smaller of the large cardinals can exist in L-like models which share many of the definability features of Gödel's L. This can be extended with forcing (via the *outer model programme*) to the strongest of large cardinals (see [3–5], for example). When GCH fails, obtaining large cardinal definability results is more of a challenge; an example is [6], which provides a definable wellorder of $H(\kappa^+)$ when the GCH fails at the measurable cardinal κ.

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In this paper we continue the study of large cardinal definability when GCH fails, by showing that there can be a normal measure on κ which is Δ_1 definable over $H(\kappa^{++})$ with $2^\kappa = \kappa^{++}$; we can even add the requirement that there is only one normal measure on κ.

1.1. Core model theory

We begin with some facts from core model theory (see [10]). Let $L[E]$ be the core model for a κ^{++}-hypermeasurable cardinal κ (i.e. a cardinal κ which is the critical point of an elementary embedding $j : V \to M$ with $H(\kappa^{++}) \subseteq M$, or equivalently, a cardinal κ which is $\kappa + 2$-strong). Then in $L[E]$ we have local definability in the following sense: For every successor cardinal $\eta \geq \omega_2$, $E \restriction \eta$ is Σ_1 definable over $H(\eta)$ with parameters, and this remains true in all cardinal-preserving set-generic extensions of $L[E]$. Using the fact that $\alpha \mapsto L_\alpha[E]$ is Σ_1 in $\langle L_\eta[E], E \restriction \eta\rangle$ and an appropriate form of condensation for substructures of $L_\eta[E]$ we may carry out the canonical construction of a $\diamondsuit$ sequence:

Fact 1.1. *In $L[E]$, for each regular cardinal α there is a $\diamondsuit_{\alpha^+}(\alpha^+ \cap \mathrm{Cof}(\alpha))$-sequence $\vec{D}_\alpha$ which is Σ_1 definable over $\langle L_{\alpha^+}[E], E \restriction \alpha^+\rangle$ in parameter $\{\alpha\}$, uniformly in α.*

Let $\langle A_i \mid i < \alpha^{++}\rangle$ be the enumeration of $P(\alpha^+)^{L[E]}$ along $<_{L[E]}$. Then $\vec{S} = \langle S_i \mid i < \alpha^{++}\rangle$ is an almost disjoint family of stationary sets where $S_i = \mathrm{Cof}(\alpha) \cap \{j < \alpha^+ \mid A_i \cap j = D_j\}$. It follows that $\vec{S}$ is Σ_1 definable with parameters over $\langle L_{\alpha^{++}}[E], E \restriction \alpha^{++}\rangle$ and therefore also over the $H(\alpha^{++})$ of any cardinal-preserving set-generic extension of $L[E]$. For technical reasons, we assume that $\vec{S}$ starts with S_{-1} instead of S_0.

For inaccessible α, let $Sacks(\alpha)$ denote the following forcing. A condition is a subset T of $2^{<\alpha}$ such that:

1. $s \in T$, $t \subseteq s \to t \in T$.
2. Each $s \in T$ has a proper extension in T.
3. If $s_0 \subseteq s_1 \subseteq \cdots$ is a sequence in T of length less than α then the union of the s_i's belongs to T.
4. Let $Split(T)$ denote the set of s in T such that both $s * 0$ and $s * 1$ belong to T. Then for some closed unbounded $C(T) \subseteq \alpha$,

$$Split(T) \supset \{s \in T \mid \mathrm{length}(s) \in C(T)\}.$$

Extension is defined by $S \leq T$ iff S is a subset of T. $Sacks(\alpha)$ is an α-closed forcing of size α^+. For $i < \alpha$, we define the i-th splitting level of T, $Split_i(T)$ to consist of all $s \in Split(T)$ such that $\{j < |s| \mid s \restriction j \in Split(T)\}$ has ordertype i. Say $S \leq_\beta T$ iff $S \leq T$ and $Split_i(T) = Split_i(S)$ whenever $i < \beta$.

For any condition S and node $s \in S$, $S|s$ denotes the subtree $\{t \in S \mid t \subseteq s \wedge s \subseteq t\}$ of S. Suppose $\langle P_\beta, \dot{Q}_\beta \mid \beta \in I\rangle$ is an iterated forcing where some of the $\dot{Q}_\beta$'s are of the form $Sacks(\alpha)^{P_\beta}$ and f is a function from such β's to $2^{<\alpha}$. Then for any condition p, we define $p|f$ by setting $(p|f)(\alpha) = p(\alpha)$ if $\alpha \notin \mathrm{dom}(f)$ and $(p|f)(\alpha) = p(\alpha)|f(\alpha)$ otherwise. We are only interested in the case when $p|f$ is also a condition in the iteration.

2. A definable normal measure

Theorem 2.1. *$Con(ZFC + \kappa$ is a κ^{++}-hypermeasurable cardinal$) \to Con(ZFC + 2^\kappa = \kappa^{++} +$ there is a normal measure U on κ such that U is Δ_1 definable over $H(\kappa^{++})$ with parameters$)$.*

We begin with an outline of the proof. We use the method of [1] to obtain a model where κ is measurable and $2^\kappa = \kappa^{++}$ using iterated κ-Sacks forcing, after a preparation below κ using iterated α-Sacks forcings for