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The quantum harmonic oscillator as a Zariski geometry

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ABSTRACT

A structure is associated with the quantum harmonic oscillator, over a fixed algebraically closed field $\mathbb{F}$ of characteristic 0, which is shown to be uncountably categorical. An analysis of definable sets is carried out, from which it follows that this structure is a Zariski geometry of dimension 1. It is non-classical in the sense that it is not interpretable in ACF_0 and in the case $\mathbb{F} = \mathbb{C}$, is not a structure on a complex manifold.

1. Introduction and background

The present paper investigates an example of physical interest (the one-dimensional quantum harmonic oscillator) using model-theoretic methods. Specifically, we associate with this system a structure QHO_N (dependent on the positive integer number N) on the universe L which is a finite cover of order N of the projective line $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{F})$, $\mathbb{F}$ an algebraically closed field of characteristic 0. We prove that QHO_N is a complete irreducible Zariski geometry of dimension 1. We also prove that QHO_N *is not classical* in the sense that the structure is not interpretable in an algebraically closed field and, for the case $\mathbb{F} = \mathbb{C}$, is not a structure on a complex manifold.

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Model-theoretic reasons for investigating QHO_N are as follows. In [2] a class of non-classical one-dimensional Zariski geometries was exhibited, by considering certain non-abelian group extensions of automorphisms of algebraic curves and constructing Zariski geometries which are acted upon by these groups via Zariski automorphisms. For a while, it looked as if these structures were the only source of non-classicality. In [10] the third author has shown that a large class of non-classical Zariski geometries can be obtained from *quantum algebras at roots of unity*, but all such geometries were of dimension (Morley rank) strictly greater than 1. The structure QHO_N is an example of a non-classical Zariski geometry of dimension 1 essentially differing from the examples in [2].

We recall briefly some physics only to provide some context for the definition of QHO_N. In the analysis of the one-dimensional quantum harmonic oscillator, one considers *momentum* and *position* operators P and Q respectively acting on a suitable Hilbert space, satisfying the so-called *canonical commutation relation*:

$$[\mathrm{Q},\mathrm{P}] = \mathrm{QP} - \mathrm{PQ} = i$$

The energy levels of this physical system are given by the eigenvalues of the *Hamiltonian* $\mathrm{H} = \frac{1}{2}(\mathrm{P}^2 + \mathrm{Q}^2)$. One can define the *creation* and *annihilation* operators, respectively:

$$\mathbf{a}^\dagger = \frac{1}{\sqrt{2}}(\mathrm{Q} - i\mathrm{P}), \qquad \mathbf{a} = \frac{1}{\sqrt{2}}(\mathrm{Q} + i\mathrm{P})$$

and these are seen to satisfy the following commutation relations

$$\left[\mathrm{H}, \mathbf{a}^\dagger\right] = \mathbf{a}^\dagger, \qquad [\mathrm{H}, \mathbf{a}] = -\mathbf{a}, \qquad \left[\mathbf{a}, \mathbf{a}^\dagger\right] = 1$$

The structure QHO_N is defined to have a quotient $\mathrm{Q}_N = (\mathcal{H}, \mathbb{F})$ which should be viewed as a bundle of eigenspaces $\mathcal{H} = \bigcup_{y \in \mathbb{P}^1(\mathbb{F})} \mathcal{H}_y$ for the *number operator* $\mathbf{N} = \mathrm{H} - \frac{1}{2}$ (thus if $y \in \mathbb{P}^1(\mathbb{F})$ then $\mathcal{H}_y$ is a one-dimensional space consisting of vectors v such that $\mathbf{N}v = yv$). This action of $\mathbf{N}$ on $\mathcal{H}$ is definable. The structure Q_N is equipped with an action of $\mathbf{a}^\dagger$ and $\mathbf{a}$ on $\mathcal{H}$, linear on each fiber, with the property that

> for each $y \in \mathbb{A}^1(\mathbb{F})$ and $v \in \mathcal{H}_y$ there is $v' \in \mathcal{H}_{y+1}$ such that $\mathbf{a}^\dagger v = bv'$ and $\mathbf{a}v' = bv$ for some b such that $b^2 = y$.

The reader will verify that the relations $[\mathbf{N}, \mathbf{a}^\dagger] = \mathbf{a}^\dagger$, $[\mathbf{N}, \mathbf{a}] = -\mathbf{a}$ and $[\mathbf{a}, \mathbf{a}^\dagger] = 1$ indeed hold.

Recall that the *first Weyl algebra* $A_1(\mathbb{F})$ is the $\mathbb{F}$-algebra generated by two generators x_1, x_2 subject to the relation $[x_1, x_2] = 1$. Although $\mathcal{H}$ is not itself a representation of $A_1(\mathbb{F})$ (only the elements x_1 and x_2 are represented, as $\mathbf{a}^\dagger$ and $\mathbf{a}$ respectively), one can obtain irreducible representations of $A_1(\mathbb{F})$ from the orbits of the additive subgroup $\mathbb{Z}$ of $\mathbb{F}$ on $\mathbb{A}^1(\mathbb{F})$ in a natural way by considering direct sums of the fibers. We record the following observation for the interested reader, though it will play only a minor role in the sequel.

Proposition 1.1. *Let $y \in \mathbb{A}^1(\mathbb{F})$. If $\{0\} \cap (y + \mathbb{Z}) = \emptyset$ then $V = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_{y+n}$ is an irreducible representation of $A_1(\mathbb{F})$. The exceptional orbit $W = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n$ decomposes as a direct sum $W^- = \bigoplus_{n \leqslant 0} \mathcal{H}_n$ and $W^+ = \bigoplus_{n > 0} \mathcal{H}_n$ of irreducible representations.*

Proof. If W' is a proper submodule, there is a minimal $|y'|$ such that $\mathcal{H}_{y'} \cap W' = \{0\}$. If $y' > 0$ then taking $v \in \mathcal{H}_{y'-1} \subseteq W'$ we obtain that $\mathbf{a}^\dagger v \in \mathcal{H}_{y'} \cap W'$. Thus $y' = 1$ (hence $y' \in \mathbb{Z}$) and $W' = W^-$. Else $y' < 0$ and given $v \in \mathcal{H}_{y'+1} \subseteq W$ we have $\mathbf{a}v = 0$, giving $W' = W^+$. □

The representation W^+ will be familiar to physicists: the eigenspaces of $\mathbf{N}$ are those giving the permissible energy values of the quantum harmonic oscillator. Typically, one arrives at these energy values by considering