Annals of Pure and Applied Logic 165 (2014) 1169–1200

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Omitting uncountable types and the strength of [0, 1]-valued logics

Xavier Caicedo [a,*], José N. Iovino [b,*]

[a] Department of Mathematics, Universidad de los Andes, Apartado Aereo 4976, Bogotá, Colombia
[b] Department of Mathematics, The University of Texas at San Antonio, One UTSA Circle, San Antonio, TX 78249-0664, USA

A R T I C L E   I N F O

Article history:
Received 15 April 2012
Accepted 27 October 2013
Available online 28 February 2014

MSC:
03C95
03C90
03B52
54E52

Keywords:
Continuous model theory
Continuous logic
Łukasziewicz logic
Łukasziewicz–Pavelka logic
Omitting types theorem

A B S T R A C T

We study a class of [0, 1]-valued logics. The main result of the paper is a maximality theorem that characterizes these logics in terms of a model-theoretic property, namely, an extension of the omitting types theorem to uncountable languages.

Introduction

In [9], Chang and Keisler introduced a model-theoretic apparatus for logics with truth values in a compact Hausdorff space whose logical operations are continuous, calling it *continuous logic*. In this paper we focus on the special case when the truth-value space is the closed unit interval $[0, 1]$. We call it *basic continuous logic*. The main result of the paper is a characterization of this logic in terms of a model-theoretic property, namely, an extension of the omitting types theorem to uncountable languages. This result generalizes a characterization of first-order logic due to Lindström [20]. By restricting basic continuous logic to particular classes of structures we obtain analogous characterizations of $[0, 1]$-valued logics that have been studied

* Corresponding authors.
E-mail addresses: xcaicedo@uniandes.edu.co (X. Caicedo), iovino@math.utsa.edu (J.N. Iovino).

0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2014.01.005

extensively, namely, Łukasziewicz–Pavelka logic [25–27] (see also Section 5.4 of [14]) and the first-order continuous logic framework of Ben Yaacov and Usvyatsov [3].

To make this more precise, consider the following logic $\mathcal{L}_0$. The semantics is given by the class of *continuous metric structures*, that is, metric spaces with uniformly continuous functions and uniformly continuous $[0,1]$-valued predicates, the distance being considered a distinguished predicate which replaces the identity relation. The sentences of $\mathcal{L}_0$ are $[0,1]$-valued and they are built as follows. The atomic formulas are the predicate symbols and the distance symbol applied to terms. The connectives are the Łukasziewicz implication ($x \to y = \min\{1-x+y, 1\}$) and the Pavelka rational constants, i.e., for each rational r in the closed interval $[0,1]$ a constant connective with value r (these are sufficient to generate, as uniform limits, all continuous connectives). The quantifiers are $\forall$ and $\exists$ are interpreted as infima and suprema of truth-values, respectively (only one of them is needed).

We observe that the restriction of $\mathcal{L}_0$ to the class of discrete metric structures is first-order logic, its restriction to the class of 1-Lipschitz structures is predicate Łukasziewicz–Pavelka logic, and its restriction to the class of complete structures yields the continuous logic framework of [3], called continuous logic in recent literature.

In general, a formula $\varphi(\bar{x})$ of an arbitrary $[0,1]$-valued logic $\mathcal{L}$ assigns to each structure M of its semantic domain and each tuple $\bar{a}$ in M a *truth-value* $\varphi^M(\bar{a})$ belonging to $[0,1]$. In this context, we may define a satisfaction relation: $M \models_{\mathcal{L}} \varphi[\bar{a}]$ if and only if $\varphi^M(\bar{a}) = 1$. Based on $\models_{\mathcal{L}}$, we introduce classical notions as consistency, semantical consequence, etc.

If λ is an uncountable cardinal and T is a theory of cardinality $\leqslant \lambda$ in a logic $\mathcal{L}$, we will say that a partial type $\Sigma(x)$ of $\mathcal{L}$ is λ-*principal* over T if there exists a set of formulas $\Phi(x)$ of cardinality $< \lambda$ such that $T \cup \Phi(x)$ is consistent and $T \cup \Phi(x) \models_{\mathcal{L}} \Sigma(x)$. The notion of ω-principal is slightly more involved (see Definition 3.4).

A logic $\mathcal{L}$ satisfies the λ-omitting types property if whenever T is a consistent theory of $\mathcal{L}$ of cardinality $\leqslant \lambda$ and $\{\Sigma_j(x)\}_{j<\lambda}$ is a set of types that are not λ-principal over T there is a model of T that omits each $\Sigma_j(x)$.

In the first part of the paper we prove the following result:

Theorem 1. $\mathcal{L}_0$ *satisfies the λ-omitting types property, for every infinite cardinal λ.*

In the second part we show that this property for uncountable cardinals characterizes $\mathcal{L}_0$:

Theorem 2. *Let $\mathcal{L}$ be a $[0,1]$-valued logic that extends $\mathcal{L}_0$ and satisfies the following properties*:

- *The λ-omitting types property for every uncountable cardinal λ,*
- *Closure under the of Łukasziewicz–Pavelka connectives* (*see below*) *and the existential quantifier,*
- *Every continuous metric structure is logically equivalent in $\mathcal{L}$ to its metric completion.*

Then every sentence in $\mathcal{L}$ is a uniform limit of sentences in $\mathcal{L}_0$.

By restricting Theorem 2 to the class of 1-Lipschitz structures we obtain a characterization of Łukasziewicz–Pavelka logic, and by restricting it to the class of complete structures we obtain an analogous characterization of continuous logic. See Corollary 4.7. However, the latter case uses a form of the λ-omitting types property asserting that the type-omitting structure is complete. This version requires a stronger notion of type principality, but it follows from the λ-omitting types property of $\mathcal{L}_0$.

Our proof of the λ-omitting types property is based on a general version of the Baire category theorem (Proposition 3.2). The proof covers at once the uncountable case and, with a minor modification, the case $\lambda = \omega$. (See Theorem 3.6.) The countable case is not new; omitting types theorems for $[0,1]$-valued logics