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Inside the Muchnik degrees II: The degree structures induced by the arithmetical hierarchy of countably continuous functions

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ABSTRACT

It is known that infinitely many Medvedev degrees exist inside the Muchnik degree of any nontrivial Π^0_1 subset of Cantor space. We shed light on the fine structures inside these Muchnik degrees related to learnability and piecewise computability. As for nonempty Π^0_1 subsets of Cantor space, we show the existence of a finite-Δ^0_2-piecewise degree containing infinitely many finite-$(\Pi^0_1)_2$-piecewise degrees, and a finite-$(\Pi^0_2)_2$-piecewise degree containing infinitely many finite-Δ^0_2-piecewise degrees (where $(\Pi^0_n)_2$ denotes the difference of two Π^0_n sets), whereas the greatest degrees in these three "finite-Γ-piecewise" degree structures coincide. Moreover, as for nonempty Π^0_1 subsets of Cantor space, we also show that every nonzero finite-$(\Pi^0_1)_2$-piecewise degree includes infinitely many Medvedev (i.e., one-piecewise) degrees, every nonzero countable-Δ^0_2-piecewise degree includes infinitely many finite-piecewise degrees, every nonzero finite-$(\Pi^0_2)_2$-countable-Δ^0_2-piecewise degree includes infinitely many countable-Δ^0_2-piecewise degrees, and every nonzero Muchnik (i.e., countable-Π^0_2-piecewise) degree includes infinitely many finite-$(\Pi^0_2)_2$-countable-Δ^0_2-piecewise degrees. Indeed, we show that any nonzero Medvedev degree and nonzero countable-Δ^0_2-piecewise degree of a nonempty Π^0_1 subset of Cantor space have the strong anticupping properties. Finally, we obtain an elementary difference between the Medvedev (Muchnik) degree structure and the finite-Γ-piecewise degree structure of all subsets of Baire space by showing that none of the finite-Γ-piecewise structures is Brouwerian, where Γ is any of the Wadge classes mentioned above.

1. Summary

1.1. Introduction

This paper is a continuation of Higuchi–Kihara [29]. Our objective in this paper is to investigate the degree structures induced by intermediate notions between the Medvedev reduction (uniformly computable

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function) and Muchnik reduction (nonuniformly computable function). We will shed light on a hidden, but extremely deep, structure inside the Muchnik degree of each Π^0_1 subset of Cantor space.

In 1963, Albert Muchnik [47] introduced the notion of Muchnik reduction as a partial function on Baire space that is decomposable into countably many computable functions. Such a reduction is also called a *countably computable* function, σ-*computable* function, or *nonuniformly computable* function. The notion of Muchnik reduction has been a powerful tool for clarifying the noncomputability structure of the Π^0_1 subsets of Cantor space [57–59,61]. Muchnik reductions have been classified in Part I [29] by introducing the notion of piecewise computability.

Remarkably, many descriptive set theorists have recently focused their attention on the concept of *piecewise definability* of functions on Polish spaces, in association with the Baire hierarchy of Borel measurable functions (see [44,45,55]). Roughly speaking, if $\mathbf{\Gamma}$ is a pointclass (in the Borel hierarchy) and $\mathbf{\Lambda}$ is a class of functions (in the Baire hierarchy), a function is said to be $\mathbf{\Gamma}$-piecewise $\mathbf{\Lambda}$ if it is decomposable into countably many $\mathbf{\Lambda}$-functions with $\mathbf{\Gamma}$ domains. If $\mathbf{\Gamma}$ is the class of all closed sets and $\mathbf{\Lambda}$ is the class of all continuous functions, it is simply called *piecewise continuous* (see for instance [32,36,46,51]). The notion of piecewise continuity is known to be equivalent to the $\mathbf{\Delta}^0_2$-measurability [32]. If $\mathbf{\Gamma}$ is the class of all sets and $\mathbf{\Lambda}$ is the class of all continuous functions, it is also called *countably continuous* [45] or σ-*continuous* [54]. Nikolai Luzin was the first to investigate the notion of countable-continuity, and today, many researchers have studied this concept, in particular, with an important dichotomy theorem (see [52,64]).

Our concepts introduced in Part I [29], such as Δ^0_2-*piecewise computability*, are indeed the lightface versions of piecewise definability. This notion is also known to be equivalent to the effective Δ^0_2-measurability [51]. See also [5,19,38] for more information on effective Borel measurability.

To gain a deeper understanding of piecewise definability, we investigate the Medvedev- and Muchnik-like degree structures induced by piecewise computable notions. This also helps us to understand the notion of relative learnability since we have observed a close relationship between lightface piecewise definability and algorithmic learning in Part I [29].

In Part II, we restrict our attention to the local substructures consisting of the degrees of all Π^0_1 subsets of Cantor space. This indicates that we consider the relative piecewise computably (or learnably) solvability of *computably-refutable problems*. When a scientist attempts to verify a statement P, his verification will be algorithmically refuted whenever it is incorrect. This *falsifiability principle* holds only when P is represented as a Π^0_1 subset of a space. Therefore, the restriction to the Π^0_1 sets can be regarded as an analogy of *Popperian learning* [11] because of the falsifiability principle. From this perspective, the universe of the Π^0_1 sets is expected to be a good playground of Learning Theory [31].

The restriction to the Π^0_1 subsets of Cantor space $2^{\mathbb{N}}$ is also motivated by several other arguments. First, many mathematical problems can be represented as Π^0_1 subsets of certain topological spaces (see Cenzer and Remmel [15]). The Π^0_1 sets in such spaces have become important notions in many branches of Computability Theory, such as *Recursive Mathematics* [23], *Reverse Mathematics* [60], *Computable Analysis* [65], *Effective Randomness* [21,49], and *Effective Descriptive Set Theory* [43]. For these reasons, degree structures on Π^0_1 subsets of Cantor space $2^{\mathbb{N}}$ are widely studied from the viewpoint of *Computability Theory* and *Reverse Mathematics*.

In particular, many theorems have been proposed on the algebraic structure of the Medvedev degrees of Π^0_1 subsets of Cantor space, such as density [13], embeddability of distributive lattices [3], join-reducibility [2], meet-irreducibility [1], noncuppability [12], non-Brouwerian property [28], decidability [16], and undecidability [56] (see also [30,57–59,61] for other properties on the Medvedev and Muchnik degree structures). The Π^0_1 sets have also been a key notion (under the name of *closed choice*) in the study of the structure of the Weihrauch degrees, which is an extension of the Medvedev degrees (see [6–8]).

Among other results, Cenzer and Hinman [13] showed that the Medvedev degrees of the Π^0_1 subsets of Cantor space are dense, and Simpson [57] questioned whether the Muchnik degrees of Π^0_1 subsets of Cantor space are also dense. However, this question remains unanswered. We have limited knowledge of