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Partial near supercompactness

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ABSTRACT

A cardinal κ is nearly θ-supercompact if for every $A \subseteq \theta$, there exists a transitive $M \models \mathrm{ZFC}^-$ closed under $<\kappa$ sequences with $A, \kappa, \theta \in M$, a transitive N, and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \theta$ and $j''\theta \in N$.[2] This concept strictly refines the θ-supercompactness hierarchy as every θ-supercompact cardinal is nearly θ-supercompact, and every nearly $2^{\theta^{<\kappa}}$-supercompact cardinal κ is θ-supercompact. Moreover, if κ is a θ-supercompact cardinal for some θ such that $\theta^{<\kappa} = \theta$, we can move to a forcing extension preserving all cardinals below θ^{++} where κ remains θ-supercompact but is not nearly θ^+-supercompact. We will also show that if κ is nearly θ-supercompact for some $\theta \geqslant 2^\kappa$ such that $\theta^{<\theta} = \theta$, then there exists a forcing extension preserving all cardinals at or above κ where κ is nearly θ-supercompact but not measurable. These types of large cardinals also come equipped with a nontrivial indestructibility result. A forcing poset is $<\kappa$-directed closed if it is γ-directed closed for all $\gamma < \kappa$ in the sense of Jech (2003) [13, Def. 21.6]. We will prove that if κ is nearly θ-supercompact for some $\theta \geqslant \kappa$ such that $\theta^{<\theta} = \theta$, then there is a forcing extension where its near θ-supercompactness is preserved and indestructible by any further $<\kappa$-directed closed θ-c.c. forcing of size at most θ. Finally, these cardinals have high consistency strength. Specifically, we will show that if κ is nearly θ-supercompact for some $\theta \geqslant \kappa^+$ for which $\theta^{<\theta} = \theta$, then AD holds in $\mathbf{L}(\mathbb{R})$. In particular, if κ is nearly κ^+-supercompact and $2^\kappa = \kappa^+$, then AD holds in $\mathbf{L}(\mathbb{R})$.

Introduction

In this paper, we introduce the nearly θ-supercompact cardinal hierarchy, a large cardinal concept inspired by [20] that stratifies the θ-supercompact cardinal hierarchy and can be made indestructible by a wide variety of forcing notions. Nearly θ-supercompact cardinals generalize embedding characterizations of weak compactness in a strong way, allowing the domains to have size θ while requiring the codomains to exhibit θ-closure under their respective elementary mappings.

Main Definition. A cardinal κ is nearly θ-supercompact if for every $A \subseteq \theta$, there exists a transitive $M \models \mathrm{ZFC}^-$ closed under $<\kappa$ sequences having the subset A and the ordinals κ and θ as elements, a transitive N, and an elementary embedding $j : M \to N$ with critical point κ such that $j(\kappa) > \theta$ and $j''\theta \in N$.

The possibilities for the partially near supercompact cardinals are diverse. The θ-supercompact cardinals are always nearly θ-supercompact, but the converse need not be true. In fact, we can have nonmeasurable nearly θ-supercompact

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[1] This article is adapted from the author's dissertation.

[2] Here, we use ZFC^- to mean the theory of ZFC without the Powerset axiom, but where ZFC is understood to be axiomatized with Collection instead of Replacement.

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cardinals κ for which θ is a weakly inaccessible cardinal above κ. Nevertheless, no degree of supercompactness necessarily implies any greater degree of near supercompactness. Moreover, nearly θ-supercompact cardinals can be made highly indestructible. Even when they possess no degree of supercompactness beyond measurability, their indestructibility can carry a consistency strength bounded below by the existence of infinitely many Woodin cardinals [14].

Main Theorem.

1. *If κ is nearly θ-supercompact for some $\theta \geqslant \kappa$ such that $\theta^{<\theta} = \theta$, then there is a forcing extension where its near θ-supercompactness is preserved and indestructible by any further $<\kappa$-directed closed θ-c.c. forcing of size at most θ.*
2. *If κ is a θ-supercompact cardinal for some θ such that $\theta^{<\kappa} = \theta$, then there exists a forcing extension where κ remains θ-supercompact but is not nearly θ^+-supercompact. Moreover, this forcing extension will preserve the cofinalities of all ordinals with cofinalities below θ^{++} or above 2^θ.*
3. *If κ is nearly θ-supercompact for some $\theta \geqslant 2^\kappa$ such that $\theta^{<\theta} = \theta$, then there exists a forcing extension preserving all cardinals and cofinalities above κ where κ is nearly θ-supercompact but not measurable.*
4. *If κ is nearly θ-supercompact for some $\theta \geqslant \kappa^+$ for which $\theta^{<\theta} = \theta$, then AD holds in $\mathbf{L}(\mathbb{R})$. In particular, if κ is nearly κ^+-supercompact and $2^\kappa = \kappa^+$, then AD holds in $\mathbf{L}(\mathbb{R})$.*

We now define recurring terminology and notation as it is used in this paper. We will be continuing to use ZFC$^-$ to mean the theory described in the footnote to the abstract. The work of Gitman, Hamkins and Johnstone in [4] shows that it is important to use the Collection axiom rather than merely Replacement when omitting the Powerset axiom in order to be able to prove consequences of ZFC such as Łoś's theorem for ultrapowers or even the fact that ω_1 is regular. Since for all η, the set $\mathbf{H}_{\eta^+}$ will model this stronger ZFC$^-$ theory axiomatized with Collection, we can still consider elementary substructures to obtain models of this theory, making the substitution innocuous. We say that a poset is $\leqslant\gamma$-directed closed if it is γ-directed closed in the sense of [13, Def. 21.6]. A poset is said to be $<\eta$-directed closed if it is γ-directed closed for all $\gamma < \eta$. Our uses of $<\nu$-closed, $\leqslant\nu$-closed, $\leqslant\nu$-distributive, $<\nu$-strategically closed, and $\leqslant\nu$-strategically closed for any cardinal ν in the context of describing properties that forcing posets possess follow the definitions from [10]. We use $<\nu$-distributive to mean $\leqslant\eta$-distributive for all $\eta < \nu$. When we use any one of these properties to describe forcing, we mean that the poset being forced over has this attribute. For sets X and Y, we use $\ell \dot{:} X \to Y$ to denote the fact that ℓ is a partial function from X into Y.

1. Near θ-supercompactness

In this section, we introduce preliminary definitions and results. Let's start with a precise definition of the main types of embeddings discussed in this paper, some observations, and alternative characterizations of near θ-supercompactness.

Definition 1.1 *(Near θ-supercompactness embedding).* An elementary embedding $j : M \to N$ with critical point κ is said to be a near θ-supercompactness embedding if it is between transitive models M and N such that $\theta \in M$, the set M is $<\kappa$-closed, $j(\kappa) > \theta$, and $j''\theta \in N$. If left unspecified, we assume that M and N are models of ZFC$^-$.

Observation 1.2.

1. *If κ is θ-supercompact, then it is nearly θ-supercompact.*
2. *If κ is nearly θ-supercompact for some $\theta \geqslant \kappa$, then κ is weakly compact and in particular inaccessible.*
3. *κ is nearly θ-supercompact if and only if it is nearly ξ-supercompact for all $\xi < \theta^+$.*
4. *If κ is nearly θ-supercompact for some $\theta \geqslant 2^{\eta^{<\kappa}}$, then κ will also be η-supercompact.*
5. *κ is nearly θ-supercompact for unboundedly many θ if and only if κ is supercompact.*

Proof. (1) follows immediately from the definition. For (2), note that if κ is nearly θ-supercompact, then it is inaccessible since it is the critical point of an embedding whose domain is a ZFC$^-$ model containing all bounded subsets of κ. It then follows that κ is weakly compact because for $\theta \geqslant \kappa$, the definition of near θ-supercompactness strengthens the weak embedding characterization of weak compactness. For the direct implication of (3), code an arbitrarily selected $A \subseteq \xi$ and a surjective function $f : \theta \to \xi$ together as a subset of θ. Then a near θ-supercompactness embedding $j : M \to N$ with critical point κ whose domain contains this coding subset witnesses the near ξ-supercompactness of κ for the arbitrarily selected $A \subseteq \xi$ because N can construct $j''\text{range}(f) = j''\xi$ from $j''\theta$ and $j(f)$. For (4), if $\theta \geqslant 2^{\eta^{<\kappa}}$, then we can code the full powerset of $P_\kappa\eta$ and all (regressive) functions from $P_\kappa\eta$ as a subset of θ. As before, let $j : M \to N$ be a near θ-supercompactness embedding with critical point κ such that the coding subset is an element of M. The induced filter on $P_\kappa\eta$ generated by using $j''\eta$ as a seed will then be an actual κ-complete normal fine measure on $P_\kappa\eta$. By seed, we simply mean any element s^* of $j(D)$ for some set $D \in M$ that induces a filter F on D defined by $A \in F \Leftrightarrow A \subseteq D$ and $s^* \in j(A)$, for every subset A of D in M. The filter measures every set in M, and thus s^* is "a seed for the M-measure F." In this case, the seed s^* is $s = j''\eta$ and D is $P_\kappa\eta$ so that $j(D) = j(P_\kappa\eta) = (P_{j(\kappa)}j(\eta))^N$. For the direct implication of (5), fix $\theta \geqslant 2^{\eta^{<\kappa}}$ for which κ is nearly θ-supercompact and hence η-supercompact by (4). $\square$