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Kripke semantics and proof systems for combining intuitionistic logic and classical logic

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ABSTRACT

We combine intuitionistic logic and classical logic into a new, first-order logic called *polarized intuitionistic logic*. This logic is based on a distinction between two dual polarities which we call *red* and *green* to distinguish them from other forms of polarization. The meaning of these polarities is defined model-theoretically by a Kripke-style semantics for the logic. Two proof systems are also formulated. The first system extends Gentzen's intuitionistic sequent calculus LJ. In addition, this system also bears essential similarities to Girard's *LC* proof system for classical logic. The second proof system is based on a semantic tableau and extends Dragalin's multiple-conclusion version of intuitionistic sequent calculus. We show that soundness and completeness hold for these notions of semantics and proofs, from which it follows that cut is admissible in the proof systems and that the propositional fragment of the logic is decidable.

1. Introduction

One of Gentzen's goals in designing the sequent calculus was to construct an *analytic* approach to proofs that could work for *both* classical and intuitionistic logic [10]. While his natural deduction proof system did not allow him to prove the *Hauptsatz* uniformly for both of these logics, his design of the sequent calculus did allow the cut-elimination theorem to be proved for both logics using the same algorithm. This early attempt at providing a *unity of logic* also presented the first demonstration of the importance of structural rules in the presentation of proof systems: in particular, the rule of contraction is not allowed on the right of Gentzen's intuitionistic sequent calculus (LJ) while it is allowed on the right in his classical sequent calculus (LK). While his approach has provided us with a common framework for the proof theory of these two logics, it did not provide us with one logic that combines classical and intuitionistic logics. Translating between and combining these logics has been repeatedly considered over the past several decades.

An important property of intuitionistic logic is its ability to embed classical logic: for an overview of several such double-negation translations by Kolmogorov, Gödel, Gentzen, and others, see [8]. This ability suggests that intuitionistic logic already contains the potential to serve as a platform for combining intuitionistic and classical reasoning. The double negation translations not only embed classical logic within intuitionistic logic but also help to explain the differences between the two.

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In Gentzen's original sequent calculi, contraction is not applicable to right-hand-side formula occurrences in intuitionistic sequents but it is available for such formula occurrences in classical sequents. One way to describe double negation translations is that they overcome this restriction in intuitionistic sequents by moving some right-hand-side formula occurrences in classical sequent proofs to negated left-hand-side formula occurrences in intuitionistic sequent proofs (where contraction is available). As has been shown by Lamarche [17] and others, a different way to see the differences between sequent calculus proofs for both classical and intuitionistic logic is to use a *one-sided* sequent calculus but with a system of *polarization*—annotations such as *input* and *output*—that distinguishes those formula occurrences that are subject to structural rules from those that are not. One might argue that the polarized approach is nothing but double-negation in disguise. However, our goal is not to see classical logic as a fragment of intuitionistic logic but rather to build proof systems and semantics in which *all connectives*—classical *and* intuitionistic—may mix freely.

An attempt to achieve such mixtures with a double negation translation must address at least the following questions:

1. If intuitionistic connectives are mixed with classical connectives that have been translated via a double-negation, how does one distinguish the parts of a formula that represent classical formulas from the parts that are just intuitionistic?
2. Even more crucially, how does one obtain cut-elimination in such a mixed setting? In the context of sequents, a double-negation represents classical formulas on the left-hand side. However, the cut rule

$$\frac{\neg A, \Gamma \vdash B \quad A, \Gamma \vdash B}{\Gamma \Gamma' \vdash B}$$

   is admissible in classical logic but *not* in intuitionistic logic. Is cut-elimination possible at all when intuitionistic and classical formulas can mix?

These questions point to the consideration of a logic in which classical connectives are added as primitives alongside intuitionistic connectives. Furthermore, it is well known that the "purely intuitionistic" connectives of implication and universal quantification exhibit characteristics that decisively distinguish them from the other connectives. In order to guarantee that these connectives do not collapse into their classical counterparts in this mixed setting, we shall rely on a polarization of connectives.

Assume for the moment that $\vee^i$ is the intuitionistic "or" that gives us the disjunction property, and that $\vee^c$ is the classical "or" that is subject to structural rules. If we are allowed to freely mix these connectives with the purely intuitionistic ones, questions arise that challenge our understanding of classical and intuitionistic logics. The two versions of disjunction would naturally give rise to two versions of *false*: assume that these are $\perp$ for $\vee^c$ and 0 for $\vee^i$. We know that $A \vee^i \neg A$ should not be provable. But what about $A \vee^c \neg A$? If negation is defined in terms of intuitionistic implication and 0 (i.e., $A \supset 0$) then the answer is *still no*. The constant 0, being associated with $\vee^i$, should not be subject to weakening. One might notice that this argument is essentially one of linear logic, and that the observations concerning structural rules do not necessarily apply in intuitionistic proof theory. In this paper, however, we shall provide a semantic explanation of the above phenomenon independently of linear logic. If we tried to explain the above non-provability in terms of a traditional double-negation translation in intuitionistic logic, we will find that neither $\neg\neg(A \vee \neg A)$ nor $\neg(A \wedge \neg A)$ are accurate translations, since they are intuitionistically provable.

To formulate a system in which the law of excluded middle can safely and transparently coexist with the disjunction property, we also require a classical notion of negation that exhibits the expected De Morgan dualities. In order to extend these dualities to a setting where arbitrary connectives may mix, there will be *dual connectives to intuitionistic implication and universal quantification*. The twin notions of negation also give rise to distinct levels of consistency: a characteristic that we explain by an enrichment of Kripke models. In particular, we shall admit *imaginary possible worlds* that may validate $\perp$ (but never 0). These models translate to Heyting algebras with an embedded boolean algebra, one that is different from the *skeleton* induced from Glivenko's transformation. We refer to this logic as *polarized intuitionistic logic* (PIL). Double negations will be crucially important in the semantic exposition of PIL, but the syntax and proof theory of PIL are independent of them.

This paper is organized as follows. Section 2 defines the syntax of formulas and their polarity assignments, without giving any meaning to these assignments. Section 3 defines the Kripke semantics for the *propositional* fragment of PIL. A translation to the Heyting algebra representation is also provided. The mixing of classical and intuitionistic quantifiers pose certain challenges. Thus we will present the first-order semantics separately in Section 5. Section 4 introduces the sequent calculus *LP* and discusses its relationship to LJ and LC. Section 6 establishes the soundness and completeness of LP (with respect to the full first-order logic). The *admissibility of cut* follows from semantic completeness. In Section 7, we present another proof system for PIL based on semantic tableau. From the correctness of this system it is also shown that the propositional fragment of PIL is decidable. In Section 8, we discuss related works, which include various double-negation translations, dual-intuitionistic logic, linear logic, polarized linear logic, LU and LC, as well as some of our own work. We summarize in Section 9.