Annals of Pure and Applied Logic 164 (2013) 112–130

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

An order-theoretic analysis of interpretations among propositional deductive systems

Ciro Russo

Departamento de Matemática, Universidade Federal da Bahia, Salvador, BA, Brazil

ARTICLE INFO

Article history:
Received 8 February 2012
Accepted 20 September 2012
Available online 27 September 2012
Communicated by J. Krajicek

MSC:
03G27
06F07

Keywords:
Consequence relation
Translation
Interpretation
Quantale

ABSTRACT

In this paper we study interpretations and equivalences of propositional deductive systems by using a quantale-theoretic approach introduced by Galatos and Tsinakis. Our aim is to provide a general order-theoretic framework which is able to describe and characterize both strong and weak forms of interpretations among propositional deductive systems also in the cases where the systems have different underlying languages.

0. Introduction

The problem of comparing logical systems can be traced back to the early twentieth century, when Brouwer introduced intuitionistic logic [6]. In the debate around the principle of excluded middle, fostered by Brouwer's ideas, a central question was whether the provable assertions of classical logic could also be formulated and proved in intuitionistic logic. Such a question led to the problem of interpreting classical logic into intuitionistic logic.

Starting from the assumption that "it is illegitimate to use the principle of excluded middle in the domain of transfinite arguments", Kolmogorov [19] proved – in 1925 – that "finitary conclusions obtained by means of the principle of excluded middle, are in fact correct and can be proved even without its use". The main result he obtained essentially asserts that any provable formula of classical logic is intuitionistically provable provided that all of its subformulas are replaced by their respective double negations. From the viewpoint of propositional calculus, the Kolmogorov interpretation is not invariant with respect to the action of substitutions: if the interpretation and a substitution are applied to a classical formula, the resulting intuitionistic formula depends on which of the two is applied first. In 1929 Glivenko [15] proved the following theorem: "An arbitrary propositional formula A is classically provable, if and only if $\neg\neg A$ is intuitionistically provable". A major difference between the two interpretations is their behavior with respect to substitutions, as the latter is invariant with respect to any substitution in the language of classical logic.

In 1934 Gentzen introduced the sequent calculi LK and LJ, for classical and intuitionistic logic respectively [14]. Such new formal systems, which gave birth to proof theory and automated deduction, offered a new perspective on the relationship

E-mail address: ciro.russo@ufba.br.

0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2012.09.006

between the two logics. Indeed the two systems differ from each other just in the type of sequents they can handle – LJ admitting exclusively sequents with a single formula or no formulas on the right-hand side (and the obvious adjustments of the rules of LK reflecting the change of the sequent-type).

The existence of algebraic semantics for certain logics can be ascribed to Lindenbaum and Tarski, who showed how it is possible to associate in a canonical way, at least at the propositional level, logical calculi (and their attendant consequence relations) with classes of algebras. We refer the reader to the article [22] for a detailed survey of these developments. Moreover, in [32], Tarski proposed two methods for attacking the *decision problem* in first-order logical systems, one of which – the so-called "indirect method" – consists in transferring the problem from a system to some other one for which it has previously been solved; in order to do that, Tarski defined the concept of translation between (first-order) logical systems.

Many years later, Blok and Pigozzi introduced the concepts of *equivalent algebraic semantics* and *algebraizable logic* [4], which requires the comparison of two consequence relations defined on different syntactic constructs (formulas, equations, sequents) and, in some cases, also on different languages. As a result of that work, the interest for interpretations and translations between logics increased rapidly in the last two decades, and many authors investigated this problem from various points of view and with different approaches; see, for instance, [3,5,8–11,13,25–27,34]. These studies have produced a wide range of concepts connected to interpretations and translations of logics.

In particular, it is shown in [13] that consequence relations (defined on sets of formulas, equations or sequents) can be represented as *structural closure operators* on quantale modules – such operators being in bijective correspondence with quotients for any given module.

The aim of this work is to study interpretations and translations between propositional logics using the algebraic techniques developed in [13,29,30]. A distinct feature of our work is the separation of the concept of a *translation* from that of an *interpretation*. Indeed, to our knowledge, since the aforementioned works by Kolmogorov, Glivenko and Tarski appeared, a translation of a language into another has only been considered as a part of an interpretation of one logical system into another – in some cases the words "interpretation" and "translation" being actually used as synonyms. On the contrary, here we will consider translations between languages as objects of study themselves, regardless – a priori – of whether an interpretation exists. A similar approach can be found in [23], where language translations are called *signature morphisms*.

Indeed, the fact that a deductive system is interpretable in another one means, roughly speaking, that the two consequence relations "agree" at some level with each other. On the other hand, the language is a syntactic object whose existence is independent from any consequence relation that, eventually, can be defined on it; indeed, we may have many different consequence relations defined on the same language. For these reasons, it is preferable that a translation between two languages regards only the connectives of the languages without any a priori involvement of the deductive apparatus of the systems. To illustrate this point, if two people speak different languages and one of them is able to translate in his own language what the other says, it does not follow that he/she also agrees with his/her interlocutor's ideas. Conversely, two people may have identical ideas but be unable to translate them into each other's language.

According to this point of view, one can encounter different situations corresponding to the existence and non-existence of a translation of languages and the interpretation of two consequence relations, and to the strengthening and weakening of the concept of interpretation.

On the other hand, logical systems with different underlying languages are, in some sense (that will appear clearer and more precise later on in the paper), objects in different categories. Therefore, a comparison between such systems requires the existence of a canonical method for putting them in the same category, namely, the existence of a suitable functor.

The main results of the paper can be briefly summarized as follows.

- In Lemma 3.3 we show that any translation between two given propositional languages $\mathcal{L}$ and $\mathcal{L}'$ induces a homomorphism between the corresponding substitution monoids. This result is completed by Theorem 3.5, in which we characterize the homomorphisms between substitution monoids that are induced by translations.
- Theorem 3.6 shows that surjectivity is a sufficient condition for a language translation to have a right-inverse translation and, therefore, to induce a monoid retraction.
- In Theorem 5.5 we extend one of the main results of [13] to interpretations and representations between deductive systems with the same language.
- In Section 6 we show that several ring-theoretic constructions can be suitably adapted to quantale modules. In particular, in Theorem 6.7, we prove that any homomorphism between two quantales defines an adjoint and co-adjoint functor (in the opposite direction) between the corresponding categories of modules. Moreover, we prove that such a functor is also a full embedding if the corresponding quantale homomorphism is surjective, and its left adjoint is a retraction of categories if the quantale homomorphism is a retraction (Theorem 6.8).
- The constructions and results of Section 6, together with the basic properties of quantales and with the results of Section 3, provide the desired functors that allow us to sensibly generalize Theorem 5.5 and several results of [13]. In particular, Theorem 7.1 and Corollary 7.2 provide the extension of Theorem 5.5 to the case of systems with different languages; Theorem 7.3 proves that, assuming the existence of a surjective translation, the characterization of Corollary 7.2 can be eased by using Theorem 3.6; last, Theorem 7.4 gives an account of weak interpretations.