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Selective covering properties of product spaces

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A B S T R A C T

We study the preservation of selective covering properties, including classic ones introduced by Menger, Hurewicz, Rothberger, Gerlits and Nagy, and others, under products with some major families of concentrated sets of reals.

Our methods include the projection method introduced by the authors in an earlier work, as well as several new methods. Some special consequences of our main results are (definitions provided in the paper):

(1) Every product of a concentrated space with a Hurewicz $\mathsf{S}_1(\Gamma, \mathrm{O})$ space satisfies $\mathsf{S}_1(\Gamma, \mathrm{O})$. On the other hand, assuming the Continuum Hypothesis, for each Sierpiński set S there is a Luzin set L such that $L \times S$ can be mapped onto the real line by a Borel function.
(2) Assuming Semifilter Trichotomy, every concentrated space is productively Menger and productively Rothberger.
(3) Every scale set is productively Hurewicz, productively Menger, productively Scheepers, and productively Gerlits–Nagy.
(4) Assuming $\mathfrak{d} = \aleph_1$, every productively Lindelöf space is productively Hurewicz, productively Menger, and productively Scheepers.

A notorious open problem asks whether the additivity of Rothberger's property may be strictly greater than $\mathrm{add}(\mathcal{N})$, the additivity of the ideal of Lebesgue-null sets of reals. We obtain a positive answer, modulo the consistency of Semifilter Trichotomy ($\mathfrak{u} < \mathfrak{g}$) with $\mathrm{cov}(\mathcal{M}) > \aleph_1$.

Our results improve upon and unify a number of results, established earlier by many authors.

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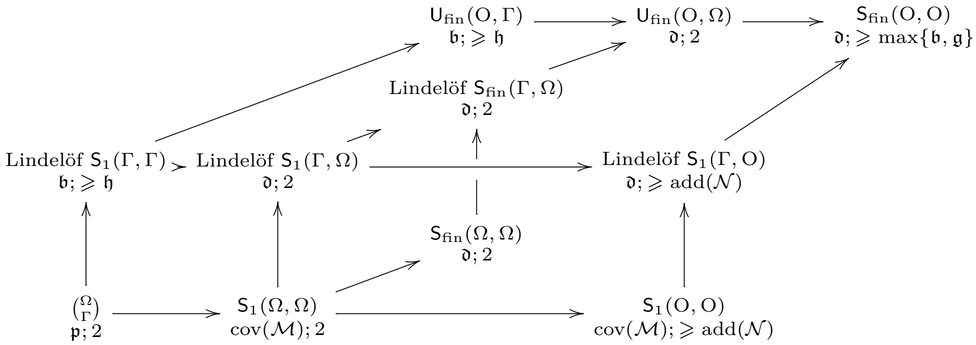

Fig. 1. The Scheepers Diagram.

1. Introduction

All topological spaces in this paper are assumed, without further mention, to be Tychonoff. Since the results presented here are new even in the case where the spaces are subsets of the real line, readers who wish to do so may assume throughout that we deal with sets of real numbers.

We study selective covering properties of products of topological spaces. Our results, that answer questions concerning classic covering properties, are best perceived in the modern framework of selection principles, to which we provide here a brief introduction.[2] This framework was introduced by Scheepers in [27] to study, in a uniform manner, a variety of properties introduced in different mathematical disciplines, since the early 1920's, by Menger, Hurewicz, Rothberger, Gerlits and Nagy, and others.

Let X be a topological space. We say that $\mathcal{U}$ is a *cover* of X if $X = \bigcup \mathcal{U}$, but $X \notin \mathcal{U}$. Often, X is considered as a subspace of another space Y, and in this case we always consider covers of X by subsets of Y, and require instead that no member of the cover contains X. Let $\mathrm{O}(X)$ be the family of open covers of X. Define the following subfamilies of $\mathrm{O}(X)$: $\mathcal{U} \in \Omega(X)$ if each finite subset of X is contained in some member of $\mathcal{U}$. $\mathcal{U} \in \Gamma(X)$ if $\mathcal{U}$ is infinite, and each element of X is contained in all but finitely many members of $\mathcal{U}$.

Some of the following statements may hold for families $\mathscr{A}$ and $\mathscr{B}$ of covers of X.

- $\binom{\mathscr{A}}{\mathscr{B}}$ Each member of $\mathscr{A}$ contains a member of $\mathscr{B}$.
- $\mathsf{S}_1(\mathscr{A},\mathscr{B})$ For each sequence $\langle \mathcal{U}_n \in \mathscr{A}\colon n \in \mathbb{N}\rangle$, there is a selection $\langle U_n \in \mathcal{U}_n\colon n \in \mathbb{N}\rangle$ such that $\{U_n\colon n \in \mathbb{N}\} \in \mathscr{B}$.
- $\mathsf{S}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$ For each sequence $\langle \mathcal{U}_n \in \mathscr{A}\colon n \in \mathbb{N}\rangle$, there is a selection of finite sets $\langle \mathcal{F}_n \subseteq \mathcal{U}_n\colon n \in \mathbb{N}\rangle$ such that $\bigcup_n \mathcal{F}_n \in \mathscr{B}$.
- $\mathsf{U}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$ For each sequence $\langle \mathcal{U}_n \in \mathscr{A}\colon n \in \mathbb{N}\rangle$, where no $\mathcal{U}_n$ contains a finite subcover, there is a selection of finite sets $\langle \mathcal{F}_n \subseteq \mathcal{U}_n\colon n \in \mathbb{N}\rangle$ such that $\{\bigcup \mathcal{F}_n\colon n \in \mathbb{N}\} \in \mathscr{B}$.

We say, e.g., that X satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ if the statement $\mathsf{S}_1(\mathrm{O}(X),\mathrm{O}(X))$ holds. This way, $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ is a property (or a class) of topological spaces, and similarly for all other statements and families of covers. In the realm of Lindelöf spaces,[3] each nontrivial property among these properties, where $\mathscr{A},\mathscr{B}$ range over O,Ω,Γ, is equivalent to one in Fig. 1 [27,14]. In this diagram, an arrow denotes implication.

[2] This introduction is adopted from [21]. Extended introductions to this field are available in [16,28,31].

[3] Indeed, all properties in the Scheepers Diagram (Fig. 1), except for those having Γ in the first argument, imply being Lindelöf.