



# Inside the Muchnik degrees I: Discontinuity, learnability and constructivism



K. Higuchi<sup>a</sup>, T. Kihara<sup>b,\*</sup>

<sup>a</sup> Department of Mathematics and Informatics, Chiba University, 1-33 Yayoi-cho, Inage, Chiba, Japan

<sup>b</sup> School of Information Science, Japan Advanced Institute of Science and Technology, Nomi 923-1292, Japan

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## ABSTRACT

Every computable function has to be continuous. To develop computability theory of discontinuous functions, we study low levels of the arithmetical hierarchy of nonuniformly computable functions on Baire space. First, we classify nonuniformly computable functions on Baire space from the viewpoint of learning theory and piecewise computability. For instance, we show that mind-change-bounded learnability is equivalent to finite  $(\Pi_1^0)_2$ -piecewise computability (where  $(\Pi_1^0)_2$  denotes the difference of two  $\Pi_1^0$  sets), error-bounded learnability is equivalent to finite  $\Delta_2^0$ -piecewise computability, and learnability is equivalent to countable  $\Pi_1^0$ -piecewise computability (equivalently, countable  $\Sigma_2^0$ -piecewise computability). Second, we introduce disjunction-like operations such as the coproduct based on BHK-like interpretations, and then, we see that these operations induce Galois connections between the Medvedev degree structure and associated Medvedev/Muchnik-like degree structures. Finally, we interpret these results in the context of the Weihrauch degrees and Wadge-like games.

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## 1. Summary

### 1.1. Introduction

Imagine the floor function, a real function that takes the integer part of an input. Although it seems easy to draw a rough graph of the floor function, it is *not* computable with respect to the standard real number representation [82], because computability automatically induces topological continuity. One way to study the floor function in computability theory is to “*computabilize*” it by changing the representation/topology of the real space (see, for instance, [83]). However, it is also important to enhance our knowledge of the

\* Corresponding author.

E-mail addresses: khiguchi@g.math.s.chiba-u.ac.jp (K. Higuchi), kihara@jaist.ac.jp (T. Kihara).

noncomputability/discontinuity level of such seemingly computable functions without changing representation/topology. Our main objective is to study low levels of the arithmetical/Baire hierarchy of functions on Baire space from the viewpoint of approximate computability/continuity and piecewise computability/continuity.

We postulate that a *nearly computable* function shall be, at the very least, *nonuniformly computable*, where a function  $f$  is said to be nonuniformly computable if for every input  $x$  there exists an algorithm  $\Psi_x$  that computes  $f(x)$  using  $x$  as an oracle, where we do not require the map  $x \mapsto \Psi_x$  to be computable. The notion of nonuniform computability naturally arises in Computable Analysis [11,88]. However, of course, most nonuniformly computable discontinuous functions are far from being computable. Then, what type of discontinuous functions are recognized as being nearly computable? A nearly computable/continuous function has to be approximated using computable/continuous functions. For instance, a Baire function appears to be *dynamically approximated* by a sequence of continuous functions and a piecewise continuous ( $\sigma$ -continuous) function appears to be *statically approximated* by countably many continuous functions.

There have been many challenges [13,83–88] in developing computability theory of (nonuniformly computable) discontinuous functions using the notion of *learnability* (dynamical approximation) and *piecewise computability* (statical approximation). Indeed, one can show the equivalence of effective learnability and  $\Pi_1^0$ -piecewise computability: the class of functions that are computable with finitely many mind changes is exactly the class of functions that are decomposable into countably many computable functions with  $\Pi_1^0$  domains. In this paper, we introduce various concepts of dynamic-approximability, and then, we characterize these concepts as static-approximability.

Now, we focus our attention on the concepts lying between (uniform) computability and nonuniform computability. In 1950–60th, Medvedev [51] and Muchnik [54] introduced the degree structure induced by uniform and nonuniform computability to formulate semantics for the intuitionistic propositional calculus based on Kolmogorov’s idea of interpreting each proposition as a problem. The degree structure induced by the Medvedev (Muchnik) reduction forms a Brouwer algebra (the dual of a Heyting algebra), where the (intuitionistic) disjunction is interpreted as the coproduct of subsets of Baire space.

Our objective is to reveal the hidden relationship between the hierarchy of nonuniformly computable functions and the hierarchy of disjunction operations. When a certain suitable disjunction-like operation such as the coproduct is introduced, we will see that one can recover the associated degree structure from the disjunction operation. As a consequence, we may understand the noncomputability feature of functions by observing the degree-theoretic behavior of associated disjunction operations. This phenomenon can be explained by using the terminology of Galois connections or adjoint functors. For instance, one can introduce a disjunction operation on Baire space using the limit-BHK interpretation of *Limit Computable Mathematics* [31] (abbreviated as LCM), a type of constructive mathematics based on Learning Theory, whose positive arithmetical fragment is characterized as Heyting arithmetic with the recursive  $\omega$ -rule and the  $\Sigma_1^0$  law of excluded middle [6,78]. Then, the “limit-BHK disjunction” includes all the information about the reducibility notion induced by learnable functions on Baire space.

Furthermore, in this paper, we introduce more complicated disjunction-like operations using BHK-like interpretations represented as “dynamic proof models” or “nested models”. For instance, a dynamic disjunction along a well-founded tree realizes the concept of learnability with ordinal-bounded mind changes, and a dynamic disjunction along an ill-founded tree realizes the concept of decomposability into countably many computable functions along a  $\Sigma_2^0$  formula.

We also interpret these results in the context of the Weihrauch degrees and Wadge-like games. We introduce a partial interpretation of nonconstructive principles including LLPO and LPO in the Weihrauch degrees and characterize the noncomputability/discontinuity level of nearly computable functions using these principles.

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