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Rudimentary and arithmetical constructive set theory

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ABSTRACT

The aim of this paper is to formulate and study two weak axiom systems for the conceptual framework of constructive set theory (CST). Arithmetical CST is just strong enough to represent the class of von Neumann natural numbers and its arithmetic so as to interpret Heyting Arithmetic. Rudimentary CST is a very weak subsystem that is just strong enough to represent a constructive version of Jensen's rudimentary set theoretic functions and their theory. The paper is a contribution to the study of formal systems for CST that capture significant stages in the development of constructive mathematics in CST.

1. Introduction

This paper is part of ongoing research to develop constructive mathematics in the conceptual framework of constructive set theory (CST). The aim is to highlight the various formal systems for CST, as weak as seems appropriate for the subject matter, in which significant mathematical topics can be developed.

1.1. Some weak axiom systems for CST

The CST conceptual framework is a set theoretical approach to constructive mathematics initiated by Myhill in [10]. It has been given a philosophical foundation via formal interpretations into versions of Martin–Löf's Intuitionistic Type Theory, [6,3,2,1]. There are several axiom systems for constructive set theory of varying logical strength. Perhaps the most familiar ones are **CZF** and $\mathbf{CZF}^+ \equiv \mathbf{CZF} + \mathbf{REA}$, see [4]. The axiom system **CZF** is formulated in the first order language $\mathcal{L}_\in$ for intuitionistic logic with equality having $\in$, an infix binary relation symbol, as the only non-logical symbol. So the logical symbols are $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$, $=$. We use the standard abbreviations for $\leftrightarrow$, $\neg$ and the bounded quantifiers $(\forall x \in t)$ and $(\exists x \in t)$. A formula is bounded if all its quantifiers are bounded.

We assume a standard axiom system for intuitionistic logic with equality. The non-logical axioms and schemes of **CZF** are the axioms of Extensionality, Emptyset, Union, Pairing and Infinity and the axiom schemes of Bounded Separation, Strong Collection, Subset Collection and Set Induction. The axiom system **CZF** is much weaker than **ZF**. Nevertheless when the law of excluded middle is added the resulting axiom system has the same theorems as **ZF**. Moreover when the Powerset Axiom and the full Separation Scheme are added an axiom system is obtained that has the same theorems as **IZF**, an axiom system that has the same logical strength as **ZF** in virtue of a double negation interpretation of **ZF** into **IZF** due to Harvey Friedman.

The main aim of this paper is to formulate and study a weak axiom system for Arithmetical CST, **ACST**, that is strong enough to represent the class *Nat* of von Neumann natural numbers and its arithmetic so that Heyting Arithmetic can be interpreted. A significant feature of **CZF** is the role of, possibly infinitary, class inductive definitions that define classes that may not be sets. We will see a similar role for finitary inductive definitions in **ACST**.

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A first approach to an axiom system for Arithmetical CST is the axiom system **BCST** + *MathInd*(*Nat*). Here (i) the axiom system **BCST** for a basic CST is obtained by leaving out from **CZF** the axiom of Infinity and the axiom schemes of Strong Collection, Subset Collection and Set Induction, while adding the axiom scheme of Replacement, and (ii) *MathInd*(*Nat*) is the axiom scheme of mathematical induction for a suitably defined class *Nat* of the von Neumann natural numbers. The axiom system **BCST** + *MathInd*(*Nat*) does not assume that *Nat* is a set. An alternative basic axiom system for arithmetic that has been considered is **ECST**, which is obtained from **BCST** by adding the axiom of Strong Infinity, the axiom that expresses the existence of the smallest inductive set, ω. In contrast to **BCST** + *MathInd*(*Nat*) the axiom system **ECST** does not have full mathematical induction, but can only derive mathematical induction for bounded formulae.

The Union Axiom and the Replacement Scheme can be combined into a single scheme, the Union-Replacement Scheme. The full strength of the Union-Replacement Scheme seems not to be needed for our purposes. It turns out that a rule of inference, the Global Union-Replacement Rule (**GURR**) can be used instead and we will see that this rule provides exactly enough power to enable definitions of the rudimentary functions on sets. The rudimentary functions were originally introduced by Ronald Jensen, see [7], in the context of classical set theory, in order to develop a good fine structure theory for Goedel's constructible sets.

So we are led to consider the very weak axiom system, **RCST**, of Rudimentary CST. This axiom system has a standard system of axioms and rules for intuitionistic logic in the language $\mathcal{L}_\in$, the rule **GURR**, the axiom of extensionality and the set existence axioms, Emptyset, Binary Intersection and Pairing for the existence of the sets $\emptyset$, $x_1 \cap x_2$, $\{x_1, x_2\}$ respectively, for sets x_1, x_2. Then **ACST** $\equiv$ $\mathbf{RCST}_0$ + *MathInd*(*Nat*) will be our preferred axiom system for Arithmetical CST, where $\mathbf{RCST}_0$ is an axiom system that has the same theorems as **RCST**, but has the advantage that it does not use any non-logical rule of inference.

Although **RCST** is very weak it is strong enough to allow the derivation of every instance of the Bounded Separation Scheme. Also each rudimentary function is a total function $V^n \to V$ on the universe of sets which can be defined by a bounded formula $\phi[x_1, \ldots, x_n, y]$ such that

$$\mathbf{RCST} \vdash (\forall x_1, \ldots, x_n)(\exists! y)\phi[x_1, \ldots, x_n, y].$$

So each rudimentary function can be given in **RCST** by a provably total single-valued class relation. It is natural to extend the language $\mathcal{L}_\in$ to a language $\mathcal{L}_\in^*$ with individual terms to represent the rudimentary functions. We are led to a simple axiom system $\mathbf{RCST}^*$ in the language $\mathcal{L}_\in^*$ which no longer needs the rule **GURR** and just has the non-logical axioms of extensionality and the term comprehension axioms for each form of term that is not a variable. We show that $\mathbf{RCST}^*$ is a conservative extension of **RCST** and we could use $\mathbf{ACST}^* \equiv \mathbf{RCST}^*$ + *MathInd*(*Nat*) as our axiom system for Arithmetical CST. As $\mathbf{ACST}^*$ is a conservative extension of **ACST** we could just as well use **ACST**. As **ACST** is in the standard language $\mathcal{L}_\in$ for set theory it is our preferred axiom system for Arithmetical CST.

1.2. Outline of paper

The paper is in two parts. Sections 2–7 form Part I on Rudimentary CST and Sections 8–10 form Part II on Arithmetical CST. Jensen's classical definition of the rudimentary functions is reviewed in Section 2 along with a classically equivalent definition that is appropriate for CST. The language $\mathcal{L}_\in^*$ and axiom system $\mathbf{RCST}^*$ are introduced in Section 3 where it is shown how the rudimentary functions are exactly the functions that can be defined by a term in $\mathbf{RCST}^*$. In Section 4 it is shown that each instance of Bounded Separation can be derived in $\mathbf{RCST}^*$. In Section 5 it is shown that every bounded formula of $\mathcal{L}_\in^*$ is equivalent in $\mathbf{RCST}^*$ to a bounded formula of $\mathcal{L}_\in$. The special case when the bounded formula is $t[x_1, \ldots, x_n] = y$ yields that the graph of each rudimentary function can be defined in $\mathbf{RCST}^*$ by a bounded formula of $\mathcal{L}_\in$. Section 6 introduces the axiom system $\mathbf{RCST}_0$, a rather useful, but unnatural axiom system for Rudimentary CST formulated in the language $\mathcal{L}_\in$. It is shown that $\mathbf{RCST}^*$ is a conservative extension of $\mathbf{RCST}_0$. The axiom system **RCST** is introduced in Section 7 and shown to have the same theorems as $\mathbf{RCST}_0$ using a result, the Term Existence Theorem for $\mathbf{RCST}^*$, whose proof has been left for another occasion.

The axiom system **ACST** = $\mathbf{RCST}_0$ + *MathInd*(*Nat*) for Arithmetical CST is introduced in Section 8 and the Finite AC Theorem is proved in Section 9 with Finitary Strong Collection derived as a corollary. The theory of finitary inductive definitions of classes is developed in Section 10.

In Section 11 we compare various axiom systems for finite set theory with weak axiom systems for set theories which have an axiom of Infinity. We have placed in Appendix A some definitions concerning the concept of an interpretation that are used in Section 11.

Part I: Rudimentary CST

2. The rudimentary functions on sets

The rudimentary functions on sets were introduced by Ronald Jensen in his famous paper [7].

The definition makes sense in any sufficiently strong axiom system for set theory. The rudimentary functions are total functions defined on the class V of all sets.