Annals of Pure and Applied Logic 164 (2013) 421–436

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Regular opens in constructive topology and a representation theorem for overlap algebras

Francesco Ciraulo

Dipartimento di Matematica, Università degli Studi di Padova, Italy

ARTICLE INFO

Article history:
Available online 3 November 2012

To Giovanni Sambin on the occasion of his 60th birthday

MSC:
03F55
54H10
06D22

Keywords:
Constructive mathematics
Regular open sets
Formal topology

ABSTRACT

Giovanni Sambin has recently introduced the notion of an overlap algebra in order to give a constructive counterpart to a complete Boolean algebra. We propose a new notion of regular open subset within the framework of intuitionistic, predicative topology and we use it to give a representation theorem for (set-based) overlap algebras. In particular we show that there exists a duality between the category of set-based overlap algebras and a particular category of topologies in which all open subsets are regular.

0. Introduction

The content of this paper can be summarized as follows: we link overlap algebras with constructive topology via the notion of regular open subset.

The definition of an overlap algebra, as given by Sambin in [10], is an intuitionistic description of the power-collection of a set. It axiomatizes not only the relation of inclusion and the operations of union and intersection, but also the binary relation, called *overlap*, which says that two subsets have an element in common. With classical logic, overlap algebras are precisely complete Boolean algebras and hence the notion of an overlap algebra is strictly stronger than that of a complete Heyting algebra.

Constructive topology is, roughly speaking, topology within an intuitionistic and predicative framework. A formal topology (see [8]) is a predicative version of an overt (or open) locale. A *positive* topology is a formal topology in which the unary positivity predicate is replaced by a positivity relation, a new notion introduced in [10].

In this paper, we propose a new definition of regular formal open subset which works both for a positive and a formal topology. We show that the collection of regular open subsets has a structure of overlap algebra and that, moreover, every (set-based) overlap algebra can be represented in this way. Finally, we give a characterization of the category of set-based overlap algebra in terms of the opposite of a particular category of positive topologies.

Sections 1 and 2 are quite of an introductory nature; they are written with the aim of making the paper as self-contained as possible. The former deals with the category of overlap algebras and its subcategory of set-based overlap

E-mail address: ciraulo@math.unipd.it.
URL: http://www.math.unipd.it/~ciraulo.

0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2012.10.006

algebras. The latter recalls from [10] the definitions of the categories of basic and positive topologies, as well as other related notions.

Section 3 contains our new definition of regular formal open subset. This is employed in the construction of a notable class of overlap algebras.

Section 4 contains, among other things, the proof that every set-based overlap algebra can be represented as the overlap algebra of regular open subsets of a suitable positive (and also formal) topology.

In Section 5 the correspondence between overlap algebras and positive topologies is extended to morphisms. In particular, it is shown that the category of set-based overlap algebras is dual to a suitable category of positive topologies.

Unless otherwise stated, all definitions and proofs of this paper are meant to be meaningful from the point of view of a *minimalist* foundational theory introduced by Maietti and Sambin in [7]. Since this theory, roughly speaking, lacks both the axiom of (unique) choice and the powerset axiom, as well as the law of excluded middle, the mathematics developed on it is valid in virtually all foundational theories such as Martin-Löf type theory, topos theory, Aczel's CZF and so on (as well as in classical mathematics).

We shall use the word "predicative" for a statement which does not require the powerset axiom (and "impredicative" for its opposite), while "constructive" will means both predicative and intuitionistic. For the sake of predicativity, we distinguish "sets", whose elements are generated by rules, as in Martin-Löf theory, and which admits some kind of induction principle, from "collections". The standard example of a collection is given by all subsets of a given set. Here a subset is essentially a predicative propositional function over a set, up to equivalence of propositions (see [11] for a constructive theory of subsets). It should be clear that a definition which uses a quantification over a collection cannot have a predicative justification. Other remarks about foundations are going to be given within the text.

Throughout this paper, X, Y, S and T will always denote sets. Moreover, we shall use x, y, z, a, b, c for elements of those sets; D, E, U, V, W will denote subsets. We write $\{a \in S \mid \varphi(a)\}$ for the subset of S corresponding to the propositional function φ. The collection of all subset of S is written $\mathcal{P}(S)$. For $U, V \subseteq S$, we put:

$$U \between V \quad \stackrel{def}{\Longleftrightarrow} \quad U \cap V \text{ is inhabited.} \tag{1}$$

Given a function $\mathcal{F}$ on $\mathcal{P}(\mathcal{S})$ and an element $a \in S$, we shall very often write $\mathcal{F}(a)$ or simply $\mathcal{F}a$ instead of $\mathcal{F}(\{a\})$. The symbols P and Q will be reserved to collections with objects p, q, r. We keep the usual symbol $\in$ for membership in a set; on the contrary, membership in a subset and in a collection will be denoted by ϵ and : (colon), respectively.

Finally, a bibliographic remark: the main source for the notions we are going to use is the still unpublished [10] (of which the author possesses a draft). However, all the basic ideas and definitions (although sometimes with different names) can be found also in other papers such as [9,3,2].

1. Overlap algebras

The notion of an overlap algebra has been recently introduced in [10] by Giovanni Sambin. It is an algebraic version of the power-collection of a set in which also the notion of "overlap", the $\between$ in Eq. (1), is axiomatized. The algebraic version of $\between$ is written $\asymp$.

Definition 1.1. An *overlap algebra* is a triple $\mathcal{P} = (P, \leqslant, \asymp)$ where $(P, \leqslant)$ is a complete lattice and $\asymp$ is a binary relation on P satisfying the following conditions:

symmetry: $p \asymp q \Longrightarrow q \asymp p$
transferring of meets: $(p \wedge r) \asymp q \Longrightarrow p \asymp (r \wedge q)$
splitting of joins: $p \asymp (\bigvee_{i\in I} q_i) \Longleftrightarrow (\exists i \in I)(p \asymp q_i)$
density: $(\forall r : P)(r \asymp p \Longrightarrow r \asymp q) \Longrightarrow p \leqslant q$

for every $p, q : P$ and every set-indexed family $\{q_i : P \mid i \in I\}$ (for I a set).

For every set S, the structure $(\mathcal{P}(S), \subseteq, \between)$ is an overlap algebra. As an example, we check that $(\forall W \subseteq S)(W \between U \Rightarrow W \between V) \Longrightarrow U \subseteq V$ (density) holds. This is easy: the antecedent gives, in particular, $(\forall a \in S)(\{a\} \between U \Rightarrow \{a\} \between V)$, that is $(\forall a \in S)(a \epsilon U \Rightarrow a \epsilon V)$; so $U \subseteq V$.

A foundational remark is needed here. We use the adjective "complete", when referred to a lattice, to mean the existence of all *set-indexed* joins and meets. This is more convenient predicatively, though coincides with usual completeness (existence of *all* joins and meets) when working within an impredicative framework.

We write 0 and 1 for the bottom and top elements of an overlap algebra, respectively. They always exists since they are the join and meet of the empty family, respectively. Sometimes, it will be convenient to assume $1 \asymp 1$. This will give $0 \neq 1$ as a consequence. In fact $0 \asymp 0$ is always false since 0 is the join of the empty family and $\asymp$ has to split joins. By the way, this same argument shows that $r \asymp 0$ is always false, whatever r is.

The next proposition characterizes overlap algebras within a classical framework. This perhaps justifies the name "algebra" for a structure which has been defined via a relational symbol.