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Relating Bishop's function spaces to neighbourhood spaces

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ABSTRACT

We extend Bishop's concept of function spaces to the concept of pre-function spaces. We show that there is an adjunction between the category of neighbourhood spaces and the category of Φ-closed pre-function spaces. We also show that there is an adjunction between the category of uniform spaces and the category of Ψ-closed pre-function spaces.

1. Introduction

In 1967, Bishop [3] proposed two approaches to topology in his constructive mathematics: one approach is based on the idea of a neighbourhood space, and the other is based on the idea of a function space. However, in his book, he did not investigate them in detail.

It turns out that neighbourhood spaces are both formal topologies, as introduced by Sambin [19–21], and constructive topological spaces (see Aczel [1]). In addition, connections between neighbourhood spaces and other constructive topological notions – in particular the Bridges–Vîţă one of an apartness space [7,9] – have been explored [14,13]. On the other hand, the approach to constructive topology based on the idea of a function space has lain relatively dormant for over forty years.

Recently, Bridges [5] has dealt with various aspects of function spaces which revive Bishop's approach to topology based on function spaces. Following Bishop [3, Definition 8, Chapter 3], we define a *function space* X to be a pair $(\underline{X}, \mathcal{F}_X)$ of a set $\underline{X}$ and a set $\mathcal{F}_X$ of functions from $\underline{X}$ to $\mathbf{R}$ satisfying the following conditions.

F1. $\mathcal{F}_X$ contains the constant functions.
F2. Sums and products of elements of $\mathcal{F}_X$ are in $\mathcal{F}_X$.
F3. The composition $\varphi \circ f$ of an element f of $\mathcal{F}_X$ and a continuous function $\varphi : \mathbf{R} \to \mathbf{R}$ is in $\mathcal{F}_X$, where $\varphi : \mathbf{R} \to \mathbf{R}$ is continuous if it is uniformly continuous on every compact interval.
F4. Uniform limits of elements of $\mathcal{F}_X$ are in $\mathcal{F}_X$; that is, if for each $\epsilon > 0$ there exists g in $\mathcal{F}_X$, with $|g(x) - f(x)| \leqslant \epsilon$ for all x in $\underline{X}$, then $f \in \mathcal{F}_X$.

Bishop called $\mathcal{F}_X$ the *topology* on $\underline{X}$.

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In this paper, we first introduce the notion of a pre-function space just as a pair of a set S and a set of real-valued functions on S, and the notion of a function space morphism according to [5]. Then we focus on the condition F3 above, and introduce the notion of a C-complete pre-function space for a set C of functions from $\mathbf{R}$ to $\mathbf{R}$; in the definition of a function space, C is taken to be the set of continuous functions in the above sense. We show that the category of C-complete pre-function spaces with function space morphisms is complete and cocomplete.

We propose a closure condition Φ_S on a set of real-valued functions on a set S, and introduce the notion of a Φ-closed pre-function space. It emerges that each Φ-closed pre-function space is a function space in Bishop's sense. Then we construct an adjunction between the category of neighbourhood spaces with continuous functions in usual sense and the category of Φ-closed pre-function spaces with function space morphisms, which relates Bishop's two approaches to topology, and show that the category of Φ-closed pre-function spaces is complete and cocomplete. We also construct an adjoint equivalence between the category of neighbourhood spaces with a compatible family of pseudometrics and the category of Φ-closed pre-function spaces.

Finally, we introduce another closure condition Ψ_S on a set of real-valued functions on a set S and the corresponding notion of a Ψ-closed pre-function space, and construct an adjunction between the category of uniform spaces with uniformly continuous functions and the category of Ψ-closed pre-function spaces with function space morphisms.

Although the results are presented in informal Bishop-style constructive mathematics [3,4,6,22,8], it is possible to formalize them in Aczel's constructive Zermelo–Fraenkel set theory (**CZF**) [2] together with the Relativized Dependent Choice (RDC).

There are other constructive treatments of topology: see, for example, Grayson [11,12].

2. Complete pre-function spaces

A *pre-function space* X is a pair $(\underline{X}, \mathcal{F}_X)$ consisting of a set $\underline{X}$ and a set $\mathcal{F}_X$ of functions from $\underline{X}$ to $\mathbf{R}$, called a *function space structure* on $\underline{X}$. According to [5], a *function space morphism* from a pre-function space X into a pre-function space Y is a mapping $f : \underline{X} \to \underline{Y}$ such that

$$\forall g \in \mathcal{F}_Y (g \circ f \in \mathcal{F}_X).$$

We write $f : X \to Y$ to denote that f is a function space morphism from X into Y, and $\mathrm{Hom}(X, Y)$ for the set of function space morphisms from X into Y.

For any set S, there are the pre-function spaces $(S, \mathbf{R}^S)$, where $\mathbf{R}^S$ is the set of functions from S into $\mathbf{R}$, and $(S, \emptyset)$, called the *discrete* function space of S and the *trivial* pre-function space of S, respectively. For each pre-function space Y, any mapping $f : S \to \underline{Y}$ is a function space morphism from the discrete function space of S into Y, and any mapping $f : \underline{Y} \to S$ is a function space morphism from Y into the trivial pre-function space of S.

Let C be a set of functions from $\mathbf{R}$ to $\mathbf{R}$ containing the identity map $\mathrm{id}_{\mathbf{R}}$ and closed under composition. A pre-function space X is C-*complete* if

$$\forall f \in \mathcal{F}_X \, \forall \varphi \in C (\varphi \circ f \in \mathcal{F}_X).$$

The discrete function spaces and the trivial pre-function spaces are C-complete for any C, and any pre-function space is $\{\mathrm{id}_{\mathbf{R}}\}$-complete. If a pre-function space X is C-complete, then X is C'-complete for any $C' \subseteq C$. Since C is closed under composition, the pre-function space $\mathbf{R}_C = (\mathbf{R}, C)$ is C-complete.

Lemma 2.1. *Let X be a pre-function space. Then*

(1) $\mathrm{Hom}(X, \mathbf{R}_C) \subseteq \mathcal{F}_X$,
(2) *X is C-complete if and only if $\mathcal{F}_X \subseteq \mathrm{Hom}(X, \mathbf{R}_C)$,*
(3) *X is C-complete if and only if $\mathcal{F}_X = \mathrm{Hom}(X, \mathbf{R}_C)$.*

Proof. Straightforward. For (1), note that $\mathrm{id}_{\mathbf{R}} \in C$. □

Specifically, $C = \mathrm{Hom}(\mathbf{R}_C, \mathbf{R}_C)$.

Proposition 2.2. *Let X be a pre-function space. Then the pre-function space $\tilde{X} = (\underline{X}, \mathrm{Hom}(X, \mathbf{R}_C))$, called the C-completion of X, is C-complete. Furthermore, $\mathrm{id}_{\underline{X}} : X \to \tilde{X}$, and if Y is a C-complete pre-function space and $f : X \to Y$, then $f : \tilde{X} \to Y$.*

Proof. Let $\varphi \in C$ and $f \in \mathrm{Hom}(X, \mathbf{R}_C)$. Then for each $\psi \in C$, since $\psi \circ (\varphi \circ f) = (\psi \circ \varphi) \circ f$ and $\psi \circ \varphi \in C$, we have $\psi \circ (\varphi \circ f) \in \mathcal{F}_X$, and therefore $\varphi \circ f \in \mathrm{Hom}(X, \mathbf{R}_C)$. Hence $\tilde{X}$ is C-complete.

Since $\mathrm{Hom}(X, \mathbf{R}_C) \subseteq \mathcal{F}_X$, by Lemma 2.1 (1), we have $\mathrm{id}_{\underline{X}} : X \to \tilde{X}$. Let Y be a C-complete pre-function space, and let $f : X \to Y$. Then for each $g \in \mathcal{F}_Y$ and $\varphi \in C$, since $\varphi \circ g \in \mathcal{F}_Y$, we have $\varphi \circ (g \circ f) = (\varphi \circ g) \circ f \in \mathcal{F}_X$, and therefore $g \circ f \in \mathrm{Hom}(X, \mathbf{R}_C)$. Hence $f : \tilde{X} \to Y$. □