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Definitional Reflection and Basic Logic

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ABSTRACT

In their *Basic Logic*, Sambin, Battilotti and Faggian give a foundation of logical inference rules by reference to certain reflection principles. We investigate the relationship between these principles and the principle of *Definitional Reflection* proposed by Hallnäs and Schroeder-Heister.

1. Introduction

Basic Logic (in short: BL) is an approach to embed various systems of propositional and quantifier logic into a framework of inferential definitions using principles of 'reflection' to generate logical rules. *Definitional Reflection* (in short: DR) is an approach to generate inference principles from clausal definitions. Both approaches not only share the term 'reflection', but show a similarity the precise kind of which will be discussed in this paper. For BL we rely on Sambin et al. [9] and Sambin [8]. DR is explained, e.g., in Hallnäs [6], Hallnäs and Schroeder-Heister [7] and Schroeder-Heister [11].[2] We confine ourselves to propositional logic in order to make the conceptual points clear.

Both BL and DR belong to the framework of proof-theoretic semantics according to which the meaning of logical constants is explained in terms of certain inference principles governing them (for an overview see [13]). Proof-theoretic semantics is normally viewed as a foundational enterprise, based on a critique of certain presuppositions of denotational semantics. However, in addition to this foundational stance, proof-theoretic semantics often claims that it is able to integrate different logics, given as deductive systems, within a single framework, which then allows to reach certain results in a uniform way. This is both true for BL and DR. For example, in the framework of BL, cut elimination is available for a variety of systems by a single method, and in the framework of DR, the rules governing a variety of logical (and also nonlogical) operators are treated uniformly in a general way. In this sense, both approaches are significant for the taxonomy

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[2] A survey of definitional reflection is subject of current work by Hallnäs and the author.

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of logical systems. In the context discussed here, we are specifically dealing with the sequent calculus and the symmetries and asymmetries inherent in this framework. A special role is played by substructural features of logical rules, especially the distinction between the additive and multiplicative association of reasoning contexts as discussed in linear logic.

2. Definitional reflection

The idea of definitional reflection is related to the program of developing general elimination rules for logical constants as presented in Schroeder-Heister [10]. Given m introduction rules for an n-ary constant c of propositional logic

$$\frac{A_1(p_1,\ldots,p_n)}{c(p_1,\ldots,p_n)} \quad \ldots \quad \frac{A_m(p_1,\ldots,p_n)}{c(p_1,\ldots,p_n)}$$

where the $A_i(p_1,\ldots,p_n)$ are premisses structured in a certain way, the elimination rules are

$$\frac{c(p_1,\ldots,p_n) \qquad \begin{matrix}[A_1(p_1,\ldots,p_n)] \\ C\end{matrix} \quad \ldots \quad \begin{matrix}[A_m(p_1,\ldots,p_n)] \\ C\end{matrix}}{C}$$

where the brackets indicate the possibility of discharging the assumption structures mentioned. In the case of conjunction and implication these rules run as follows:

$$\frac{p_1 \qquad p_2}{p_1 \wedge p_2} \qquad \frac{p_1 \wedge p_2 \qquad \begin{matrix}[p_1 \quad p_2] \\ C\end{matrix}}{C} \qquad \frac{\begin{matrix}[p_1] \\ p_2\end{matrix}}{p_1 \to p_2} \qquad \frac{p_1 \to p_2 \qquad \begin{matrix}[p_1 \Rightarrow p_2] \\ C\end{matrix}}{C}$$

In the E-rules the premiss structures occur in assumption position. There the fact that premisses may depend on assumptions is represented by 'rules of higher levels', i.e., by using some sort of 'structural' implication '⇒'.[3] The idea behind this approach is that the I-rules represent a kind of 'definition' of c, and the E-rule says that everything that follows from each defining condition A_i of c follows from c itself (for simplicity, we omit the arguments $p_1,\ldots,p_n$). The E-rule is called a rule of 'definitional reflection', since it expresses an act of 'reflection' on the inferential definition given by the I-rules.

More generally, the idea of definitional reflection was proposed by Hallnäs for a definitional system of clauses of the form

$$a \Leftarrow B$$

in the style of an inductive definition or of a logic program. Here a is an atom and B a potentially complex condition formulated in some logical or structural system. In the simplest case B may just be a list of atoms, in more complicated cases one may consider expressions in first-order or even higher-order logic or some fragment thereof. The relationship to logic programming (which includes resolution-based evaluation mechanisms) has been investigated by Hallnäs and Schroeder-Heister [7], and is also the background of certain extensions of logic programming. Formulated in a sequent-style framework, a definition of a consisting of clauses

$$\begin{cases} a \Leftarrow B_1 \\ \quad \vdots \\ a \Leftarrow B_n \end{cases}$$

leads to right and left introduction rules of the following form:

$$\frac{\Gamma \vdash B_i}{\Gamma \vdash a}\,(\vdash a) \qquad \frac{\Gamma, B_1 \vdash C \quad \ldots \quad \Gamma, B_n \vdash C}{\Gamma, a \vdash C}\,(a \vdash) \tag{1}$$

where the left-rules are called rules of 'definitional closure' and the right-rule is called 'definitional reflection'. Whereas the closure rules are local rules representing reasoning along the clauses of the given definition, the reflection rule is of a more global kind: It reflects upon the definition as a whole, saying that everything that can be obtained from each defining condition of a, can be obtained from a itself. If variables are present in a, these rules becomes more complicated, with certain provisos to be respected in their application. Also, there is no restriction in the given definition to be monotone or well-founded, so, as in logic programming, the programmer has full freedom to formulate a definition. In this sense DR is much more general than generalized elimination rules for logical constants, which are a special application of it. However, the philosophical idea is much the same: Taking all definientia of a definition together, then the definiendum expresses what these definitientia have in common, i.e., their 'common content'. Definitional reflection has at its target this common content.

[3] We do not discuss here ways of avoiding higher-level rules, which in standard logic is possible, though not in the general case. See [14] for details.