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Cantor theorem and friends, in logical form

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ABSTRACT

We prove a generalization of the hyper-game theorem by using an abstract version of inductively generated formal topology. As applications we show proofs for Cantor theorem, uncountability of the set of functions from $\mathcal{N}$ to $\mathcal{N}$ and Gödel theorem which use no diagonal argument.

1. Introduction

In the logical literature there are many results which share a common flavor, namely, the use of a diagonal argument. Let us recall here just few examples: Cantor theorem, which states that there is no surjective function from a set A into the collection of the subsets of A, uncountability of the set of functions from $\mathcal{N}$ to $\mathcal{N}$, which, given any map f from $\mathcal{N}$ to $\mathcal{N} \to \mathcal{N}$ shows how to provide a function that cannot be in the image of f, Gödel theorem, which, given any sufficiently expressive, effectively axiomatizable and consistent theory of the natural numbers, shows that there are propositions which are independent from such a theory.

We want to propose here a different approach to these results which uses no diagonal argument and meanwhile makes explicit what are the common ideas behind all of them. Indeed, we will show how to prove them by using an abstract version of inductively generated formal topology that we are going to introduce in the next section.

2. Abstract proof systems

We introduce here *abstract proof systems* as a variant of the notion of inductively generated formal topologies. Then we will show how some standard results on inductively generated formal topologies apply also to abstract proof systems (for a more detailed account on inductively generated formal topology the reader is invited to look at [4] or [8]).

Definition 2.1 *(Abstract proof systems).* An *abstract proof system* is a triple (A, I, C) such that A is a set, $I(a)$ is a set for any $a \in A$ and $C(a, i)$ is a subset of A for any $a \in A$ and $i \in I(a)$.

In the following we will say that the couple $I(-), C(-,-)$ is the axiom-set of the abstract proof system (A, I, C).

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The name of *abstract proof system* for the triple above is due to the fact that in any abstract proof system (A, I, C) it is possible to define an infinitary relation, namely, the *entailment relation* $\lhd$, by using the following inductive rules

$$\text{(reflexivity)}\quad \frac{a \in U}{a \lhd U} \qquad \text{(infinity)}\quad \frac{i \in I(a) \quad (\forall y \in C(a,i))\ y \lhd U}{a \lhd U}$$

and then it is sound to read $a \lhd U$ as "a is derivable from assumptions in U".

The connection between abstract proof systems and inductively generated formal topologies is immediate. Indeed, the definition is almost the same except for the lack of a condition guaranteeing distributivity of the finite intersection of open subsets over arbitrary unions. Thus, also for abstract proof systems one can propose a topological interpretation, namely, the elements of A and the entailment relation can be interpreted respectively into the open subsets of a topological space and its coverage relation. Indeed, in this case, one immediately obtains that if $a \lhd U$ holds then the interpretation of a is covered by the open determined by the union of the open subsets where the elements of U are interpreted provided the interpretation satisfies the axioms, namely, for any $a \in A$ and $i \in I(a)$, the interpretation of a is covered by the interpretation of $C(a, i)$ (in fact, not only this interpretation is valid, but it is possible to prove classically that it is also complete when a countable quantity of axioms is considered [8]).

However the entailment relation of an abstract proof system is not a very satisfactory description of the topological situation since it does not allow to express many aspects which are relevant from a topological point of view. Hence, in order to provide a constructive approach to topology it is better to work within a richer framework like the one proposed in [4].

We borrow here for abstract proof systems a result on inductively generated formal topologies that we are going to use in the following (see [4]).

Theorem 2.2. *Let (A, I, C) be an abstract proof system and $\lhd$ its entailment relation. Then the following conditions are admissible.*

$$(\textit{axiom cond.})\quad \frac{i \in I(a)}{a \lhd C(a,i)} \qquad (\textit{transitivity})\quad \frac{a \lhd U \quad (\forall u \in U)\ u \lhd V}{a \lhd V}$$

Proof. The *axiom condition* is straightforward since by *reflexivity* we have that, for any $y \in C(a, i)$, $y \lhd C(a, i)$ holds and hence the result follows by *infinity*.

On the other hand *transitivity* requires a proof by induction on the length of the derivation of $a \lhd U$. Now, if $a \lhd U$ because $a \in U$ then $a \lhd V$ follows by logic since we are assuming that, for all $u \in U$, $u \lhd V$. On the other hand, if $a \lhd U$ because there exists some $i \in I(a)$ such that, for all $y \in C(a, i)$, $y \lhd U$, then by inductive hypothesis, $y \lhd V$ and hence we can conclude $a \lhd V$ by *infinity*. □

Apart for the previous result, we are going to use only another general result on abstract proof systems.

Theorem 2.3. *Let (A, I, C) be an abstract proof system and suppose that $a \lhd U$. Then if, for all $i \in I(a)$, $a \in C(a, i)$ then $a \in U$.*

Proof. The proof is by induction on the length of the derivation of $a \lhd U$. Now, if $a \lhd U$ has been derived by *reflexivity* then $a \in U$ holds and hence the thesis follows trivially by logic. On the other hand, if an instance of *infinity* has been used then there exists some index $i^* \in I(a)$ such that, for all $y \in C(a, i^*)$, $y \lhd U$ holds. Hence, by inductive hypothesis, we get that if, for all $j \in I(y)$, $y \in C(y, j)$ then $y \in U$. Suppose now that for all $i \in I(a)$, $a \in C(a, i)$; then, in particular, we get that $a \in C(a, i^*)$ and hence we can conclude by logic that if, for all $j \in I(a)$, $a \in C(a, j)$ then $a \in U$. So we conclude $a \in U$ by using again the assumption that, for all $i \in I(a)$, $a \in C(a, i)$ holds. □

Till now we worked with the general notion of abstract proof system but in the following we will consider only a special case, namely, *singleton abstract proof systems*. These are abstract proof systems such that, for each $a \in A$, there is exactly one axiom, namely, the set of indexes $I(a)$ is a singleton.

For sake of a simpler notation, when considering singleton abstract proof systems we will omit any reference to the set of the indexes and its elements and we will say that (A, C) is a singleton abstract proof system if A is a set, and $C(a)$ is a subset of A for any $a \in A$. Of course, also the rules to generate the entailment relation are simplified in the obvious way, that is,

$$\text{(reflexivity)}\quad \frac{a \in U}{a \lhd U} \qquad \text{(infinity)}\quad \frac{(\forall y \in C(a))\ y \lhd U}{a \lhd U}$$

Then we have the following corollary as a consequence of Theorem 2.3.

Corollary 2.4. *Let (A, C) be a singleton abstract proof system and suppose that $a \lhd U$. Then $a \in C(a)$ yields $a \in U$. In particular, if $a \lhd \emptyset$ then $a \notin C(a)$.*