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Non-forking frames in abstract elementary classes

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ABSTRACT

The stability theory of first order theories was initiated by Saharon Shelah in 1969. The classification of abstract elementary classes was initiated by Shelah, too. In several papers, he introduced non-forking relations. Later, Shelah (2009) [17, II] introduced the good non-forking frame, an axiomatization of the non-forking notion.

We improve results of Shelah on good non-forking frames, mainly by weakening the stability hypothesis in several important theorems, replacing it by the almost λ-stability hypothesis: The number of types over a model of cardinality λ is at most λ^+.

We present conditions on K_λ, that imply the existence of a model in $K_{\lambda^{+n}}$ for all n. We do this by providing sufficiently strong conditions on K_λ, that they are inherited by a properly chosen subclass of K_{λ^+}. What are these conditions? We assume that there is a 'non-forking' relation which satisfies the properties of the non-forking relation on superstable first order theories. Note that here we deal with models of a fixed cardinality, λ.

While in Shelah (2009) [17, II] we assume stability in λ, so we can use brimmed (= limit) models, here we assume almost stability only, but we add an assumption: The conjugation property.

In the context of elementary classes, the superstability assumption gives the existence of types with well-defined dimension and the ω-stability assumption gives the existence and uniqueness of models prime over sets. In our context, the local character assumption is an analog to superstability and the density of the class of uniqueness triples with respect to the relation $\preccurlyeq_{bs}$ is the analog to ω-stability.

1. Introduction

The book [15], on elementary classes, i.e., classes of first order theories, presents properties of theories, which are so-called 'dividing lines' and investigates them. When such a property is satisfied, the theory is low, i.e., we can prove structure theorems, such as:

(1) The fundamental theorem of finitely generated Abelian groups.
(2) Artin–Wedderburn Theorem on semi-simple rings.
(3) If V is a vector space, then it has a basis B, and V is the direct sum of the subspaces $span\{b\}$ where $b \in B$.

(We do not assert that these results follow from the model theoretic analysis, but they merely illustrate the meaning of 'structure'.) But when such a property is not satisfied, we have *non-structure*, namely, there is a witness that the theory is complicated, and there are no structure theorems. This witness can be the existence of many models in the same power.

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There has been much work on classification of elementary classes, and some work on other classes of models.

The main topic in the recently published book, [17], is *abstract elementary classes* (in short AEC). There are two additional books which deal with AEC's ([1] and [6]).

From the viewpoint of the algebraist, model theory of first order theories is somewhat close to universal algebra. But he prefers focusing on the structures, rather than on sentences and formulas. Our context, abstract elementary classes, is closer to universal algebra, as our definitions do not mention sentences or formulas.

As superstability is one of the better dividing lines for first order theories, it is natural to generalize this notion to AEC's. A reasonable generalization is that of the existence of a good λ-frame (see Definition 2.1.1), introduced in [17, II]. In [17, II] we assume existence of a good λ-frame and either get a non-structure property (in λ^{++}, at least where $2^{\lambda} < 2^{\lambda^+} < 2^{\lambda^{++}}$) or derive a good λ^+-frame from it.

The main tool in studying superstability is the independence relation, so-called 'non-forking'. So let us discuss the issue of independence.

"In the 1930's, van der Waerden [van der Waerden 1949] and Whitney [Whitney 1935] abstracted the following properties of linear independence in vector spaces and algebraic independence in fields and used them to define the general notion of an independence relation" [2]. Let us describe van der Waerden's notion in terms of an element a depending on a set X:

(1) (Reflexivity) a depends on $\{a\}$.
(2) (Monotonicity) If a depends on X and $X \subseteq Y$ then a depends on Y.
(3) (Transitivity) If a depends on X and each $x \in X$ depends on Y then a depends on Y.
(4) (Exchange axiom) If a depends on $X \cup \{b\}$ but a does not depend on X then b depends on $X \cup \{a\}$.
(5) (Finite character) If a depends on X then a depends on a finite subset of X.

The notion of forking (in the context of first order theories) also specializes to linear independence and algebraic independence. It is not, strictly speaking, a generalization of the usual notion, since it is stronger in some respects, weaker in others. However, it retains the most important consequence of the theory, the ability to assign a dimension to each member of certain classes of models (see [11]).

In stability theory of first order theories we deal with a ternary relation, 'non-forking', which intuitively means 'A is free from B relative to C'. Baldwin [2] presents three differences between this notion and the standard one:

(1) In stability theory of first order theories the transitivity of dependence fails, but we have transitivity of independence: If 'A is free from B relative to C' and '$A \cup B$ is free from D relative to B', then 'A is free from D relative to C'.
(2) The element a is replaced by a set A. Since a singleton is a set, in this sense we generalize the independence relation.
(3) In stability theory we define a is independent from X over A instead of only over empty set and study what happens when A changes.

Here we deal with a much more general case: Abstract elementary classes (in short AEC's). If we consider the study of first order theory T as the study of the class of models $\{M\colon\ M \models T\}$, then the context of abstract elementary classes is a generalization of that of first order theories. There are well-known theorems on first order theories, that are wrong or very hard to prove in the context of AEC's. The main reason is that the Compactness Theorem fails. Concerning AEC's see Section 1.

Shelah defines in [17, II] a set of axioms, which a non-forking relation should satisfy, in the context of AEC. An AEC with a non-forking relation that satisfies this set of axioms is called 'a good frame'. This non-forking relation deals essentially with an element and a model. [Actually it is a relation on quadruples (M_0, M_1, a, M_3) which intuitively means 'a is free from M_1 relative to M_0' (M_3 is an ambient model, which is needed in the AEC context, because we cannot use a monster model as in the stability theory for first order theories).]

Until this point we have spoken about the following independence notions:

(1) The standard: between an element and a set.
(2) Non-forking in the context of first order theories: essentially between sets.
(3) Axioms for a non-forking relation on AEC's: essentially between an element and a model.

The current work is a generalization of [17, II]. We replace the stability assumption by the almost stability assumption, categoricity in λ and the conjugation property. We define a semi-good λ-frame as a good λ-frame minus stability in λ with almost stability in λ.

A note about the hypotheses: When we write a hypothesis, we assume it until we write another hypothesis, but usually we recall the hypothesis at the beginning of the following section. Sometimes we write 'but we do not use local character'. It is important to write this because we want to apply theorems we prove here, in papers, in which local character is not assumed (for example [10]). For the same reason, in Hypothesis 3.0.9 we assume weak assumptions.