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Duality, projectivity, and unification in Łukasiewicz logic and MV-algebras

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ABSTRACT

We prove that the unification type of Łukasiewicz (infinite-valued propositional) logic and of its equivalent algebraic semantics, the variety of MV-algebras, is nullary. The proof rests upon Ghilardi's algebraic characterisation of unification types in terms of projective objects, recent progress by Cabrer and Mundici in the investigation of projective MV-algebras, the categorical duality between finitely presented MV-algebras and rational polyhedra, and, finally, a homotopy-theoretic argument that exploits lifts of continuous maps to the universal covering space of the circle. We discuss the background to such diverse tools. In particular, we offer a detailed proof of the duality theorem for finitely presented MV-algebras and rational polyhedra—a fundamental result that, albeit known to specialists, seems to appear in print here for the first time.

1. Introduction

The origins of the theory of unification are usually traced back to the doctoral thesis that Herbrand defended at the Sorbonne in the summer of 1930; an annotated English translation of its final Chapter 5 is available in [33, pp. 525–581]. It was only with Robinson's landmark paper [28] on resolution, however, that the first unification algorithm with termination and correctness proofs appeared in print. Unification has attracted continuing interest to this day as a basic tool in automated deduction. The study of unification modulo an equational theory that grew out of such pioneering works as [27] has acquired increasing significance in recent years; see the extensive survey [3], and also [2] for a more recent survey focused on modal logic. The classical, syntactic unification problem is: given two terms s, t (built from function symbols and variables), find a *unifier* for them, that is, a uniform replacement of the variables occurring in s and t by other terms that makes s and t identical. When the latter syntactical identity is replaced by equality modulo a given equational theory E, one speaks of E-*unification*. Unsurprisingly, E-unification can be far harder than syntactic unification even when the theory E comes from the least exotic corners of the mathematical world. For instance, it may well be impossible to uniformly decide whether two terms admit at least one unifier, i.e. whether they are *unifiable* at all; and even when

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the two terms indeed are unifiable, there may well be no most general unifier for them, contrary to the situation in the syntactic case. In light of these considerations, perhaps the most basic piece of information one would like to have about E in connection with unification issues is its *unification type*.[1] In order to define it precisely, let us recall some standard notions.

We consider a set $\mathscr{F}$ of function symbols—the *signature*—along with a further set $\mathscr{V}$—the *variables*; each function symbol comes with its own arity, an integer $n \geqslant 0$, with $n = 0$ being included to accommodate constants. It is usual to require that $\mathscr{V}$ be countable. Let us therefore set, once and for all throughout the paper,

$$\mathscr{V} = \{X_1, X_2, \ldots\}.$$

We then let $\mathrm{Term}_{\mathscr{V}}(\mathscr{F})$ be the *term algebra* built from $\mathscr{F}$ and $\mathscr{V}$ in the usual manner [6, Definition 10.1]. A *substitution*[2] is a mapping $\sigma : \mathscr{V} \to \mathrm{Term}_{\mathscr{V}}(\mathscr{F})$ that acts identically to within a finite number of exceptions, i.e. is such that $\{X \in \mathscr{V} \mid \sigma(X) \neq X\}$ is a finite set. Substitutions compose in the obvious manner. By an *equational theory* over the signature $\mathscr{F}$ one means a set $E = \{(l_i, r_i) \mid i \in I\}$ of pairs of terms $l_i, r_i \in \mathrm{Term}_{\mathscr{V}}(\mathscr{F})$, where I is some index set. The set of equations E axiomatises the *variety of algebras* [6, Theorem 11.9] consisting of the models of the theory E, written $\mathbb{V}_E$.

Now a (*symbolic*) *unification problem modulo* E is a finite set of pairs

$$\mathscr{E} = \big\{(s_j, t_j) \,\big|\, s_j, t_j \in \mathrm{Term}_{\mathscr{V}}(\mathscr{F}),\ j \in J\big\},$$

for some finite index set J. A *unifier* for $\mathscr{E}$ is a substitution σ such that

$$E \models \sigma(s_j) \approx \sigma(t_j),$$

for each $j \in J$, i.e. such that the equality $\sigma(s_j) = \sigma(t_j)$ holds in every algebra of the variety $\mathbb{V}_E$ in the usual universal-algebraic sense [6, p. 78]. The problem $\mathscr{E}$ is *unifiable* if it admits at least one unifier. The set $U(\mathscr{E})$ of unifiers for $\mathscr{E}$ can be partially ordered as follows. If σ and τ are substitutions and $V \subseteq \mathscr{V}$ is a set of variables, we say that σ is *more general*[3] than τ (with respect to E and V), written $\tau \preccurlyeq_E^V \sigma$, if there exists a substitution ρ such that

$$E \models \tau(X) \approx (\rho \circ \sigma)(X)$$

holds for every $X \in V$. This amounts to saying that τ is an instantiation of σ, but only to within E-equivalence, and only as far as the set of variables V is concerned. We endow $U(\mathscr{E})$ with the relation $\preccurlyeq_E^V$, where V is the set of variables occurring in the terms s_j, t_j with $(s_j, t_j) \in \mathscr{E}$, as j ranges in J. The relation $\preccurlyeq_E^V$ is a pre-order. There is an equivalence relation $\sim$ on $U(\mathscr{E})$ that identifies σ and τ if and only if $\tau \preccurlyeq_E^V \sigma$ and $\sigma \preccurlyeq_E^V \tau$ both hold. The quotient set $\frac{U(\mathscr{E})}{\sim}$ carries the canonical partial order $\leqslant_E^V$ associated to the pre-order $\preccurlyeq_E^V$; by definition, $[\sigma] \leqslant_E^V [\tau]$ if and only if $\sigma \preccurlyeq_E^V \tau$, where $[\sigma]$ and $[\tau]$ respectively denote the equivalence classes induced by $\sim$ of the unifiers σ and τ. We call $(\frac{U(\mathscr{E})}{\sim}, \leqslant_E^V)$ the *partially ordered set of unifiers for* $\mathscr{E}$, even though its elements actually are equivalence classes of unifiers.

The (*symbolic*) *unification type* of the unification problem $\mathscr{E}$ is:

- *unitary*, if $\leqslant_E^V$ admits a maximum $[\mu] \in \frac{U(\mathscr{E})}{\sim}$;
- *finitary*, if $\leqslant_E^V$ admits no maximum, but admits finitely many maximal elements $[\mu_1], \ldots, [\mu_u] \in \frac{U(\mathscr{E})}{\sim}$ such that every $[\sigma] \in \frac{U(\mathscr{E})}{\sim}$ lies below some $[\mu_i]$;
- *infinitary*, if it is not finitary, and $\leqslant_E^V$ admits infinitely many maximal elements $\{[\mu_i] \in \frac{U(\mathscr{E})}{\sim} \mid i \in I\}$, for I an infinite index set, such that every $[\sigma] \in \frac{U(\mathscr{E})}{\sim}$ lies below some $[\mu_i]$;
- *nullary*, if none of the preceding cases applies.

It is understood that the list above is arranged in decreasing order of desirability. In the best, unitary case, any element of the maximum equivalence class $[\mu]$ is called a *most general unifier* for $\mathscr{E}$, or *mgu* for short. An mgu is then unique up to the relation $\sim$, whence one speaks of *the* mgu for $\mathscr{E}$. If $[\mu]$, on the other hand, is maximal but not a maximum, then any element of $[\mu]$ is called a *maximally general unifier*.

The *unification type* of the equational theory E is now defined to be the worst unification type occurring among the unification problems $\mathscr{E}$ modulo E.

This paper is devoted to an investigation of the unification type of *Łukasiewicz* (*infinite-valued propositional*) *logic*, a non-classical system going back to the 1920's (cf. the early survey [20, §3], and its annotated English translation in

[1] Strictly speaking, throughout this paper we are concerned with the *elementary unification type of* E, meaning that in unification problems and unifiers we do not allow terms with additional function symbols not included in the signature $\mathscr{F}$ of E; see [3, Definition 3.9].

[2] It would be more expedient to define unifiers for $\mathscr{E}$ as substitutions having a finite domain coincident with the set of variables occurring in $\mathscr{E}$. This would perfectly match the definition of the pre-order $\preccurlyeq_E^V$ on unifiers recalled below, which only compares unifiers on those variables occurring in $\mathscr{E}$. We have nonetheless chosen to follow [3] in the basic definitions in order not to depart from established practice; cf. [3, 3.2.1] for a related discussion.

[3] The convention adopted in [3] is that '$\tau \preccurlyeq_E^V \sigma$' means '$\tau$ is more general than σ', whereas here we are following [14] in choosing the opposite reading.