Annals of Pure and Applied Logic 164 (2013) 211–229

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Locally o-minimal structures and structures with locally o-minimal open core

Antongiulio Fornasiero

Seconda Università di Napoli, Viale Lincoln 5, 81100 Caserta, Italy

ARTICLE INFO

Article history:
Received 23 May 2011
Received in revised form 24 September 2012
Accepted 3 October 2012
Available online 26 October 2012
Communicated by A.J. Wilkie

MSC:
primary 03C64
secondary 12J15

Keywords:
O-minimal
Locally o-minimal
Open core
Definably complete
Baire

ABSTRACT

We study first-order expansions of ordered fields that are definably complete, and moreover either are locally o-minimal, or have a locally o-minimal open core. We give a characterisation of structures with locally o-minimal open core, and we show that dense elementary pairs of locally o-minimal structures have locally o-minimal open core.

1. Introduction

We study locally o-minimal structures and structures with locally o-minimal open core. All structures considered will be definably complete expansions of ordered fields: structures which are locally o-minimal but not definably complete (e.g., weakly o-minimal structures), while important and interesting, are outside the scope of this article; we refer the reader interested in locally o-minimal structures which are not definably complete to [20].

One of the natural generalisations of o-minimality (for definably complete structures) is requiring that every unary set definable (with parameters) is *locally* a finite union of points and intervals.

Definition 1.1. $\mathbb{K}$ is **locally o-minimal** if, for every $X \subset \mathbb{K}$ definable, and for every $x \in \mathbb{K}$, there exists $y > x$ such that, either $(x, y) \subseteq X$, or $(x, y) \subseteq \mathbb{K} \setminus X$.

Much of the theory of o-minimal structures can be generalised to locally o-minimal ones: for instance, a version of the Monotonicity Theorem (Section 5.1; see also [19]) and Miller's dichotomy (Theorem 5.18) hold also for locally o-minimal structures; moreover, a weak version of cell decomposition holds (Theorem 5.6; see also [19] for a different kind of cell decomposition); besides, A. Pillay's theorem that a group definable in an o-minimal structure can be equipped with a definable

E-mail address: antongiulio.fornasiero@gmail.com.
URL: http://www.dm.unipi.it/~fornasiero/.

0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2012.10.002

topology making it a topological group can be extended to locally o-minimal structures (Theorem 5.25). In Section 3.1 and Section 5 we will study locally o-minimal structures more in details. One advantage of locally o-minimal structures over o-minimal ones is that they form an elementary class: that is, an ultraproduct of locally o-minimal structures is locally o-minimal.

A further generalisation of local o-minimality is given by structures with locally o-minimal open core.

Definition 1.2. The **open core** of $\mathbb{K}$ is the reduct of $\mathbb{K}$ generated by all definable open subsets of $\mathbb{K}^n$, for every $n \in \mathbb{N}$.

Definition 1.3. $\mathbb{K}$ has locally o-minimal open core if the structure generated by all open definable sets is locally o-minimal.

The main results of this article can be summed up in the following two theorems.

Theorem A. *(See Theorem 3.3.) The open core of $\mathbb{K}$ is locally o-minimal iff every definable discrete subset of $\mathbb{K}$ is bounded.*

In [6, §9] we defined d-minimal topological structures, and we proved some results about the theory of dense elementary pairs of such structures (see Definition 6.2). We will show that locally o-minimal structures are d-minimal topological structures; thus, we will be able to apply the results in [6] to the present situation, and we will spell out some consequences.

Given $d \leqslant n \in \mathbb{N}$, we denote by $\prod_d^n : \mathbb{K}^n \to \mathbb{K}^d$ the projection onto the first d coordinates. Given $X \subseteq \mathbb{K}^{n+m}$ and $\bar{c} \in \mathbb{K}^n$, we define $X_{\bar{c}} := \{\bar{y} \in \mathbb{K}^m \colon \langle \bar{c}, \bar{y} \rangle \in X\}$.

Definition 1.4. (See [6, 9.1].) Let $\mathbb{K}^* \succcurlyeq \mathbb{K}$ be ω-saturated. $\mathbb{K}$ is a **d-minimal topological structure** if:

(DM1) For every $X \subseteq \mathbb{K}^*$ definable (with parameters in $\mathbb{K}^*$), if X has empty interior, then X is a finite union of discrete sets;

(DM2) For every $X \subseteq \mathbb{K}^n$ definable and discrete, $\prod_1^n(X)$ has empty interior;

(DM3) Given $X \subseteq \mathbb{K}^2$ and $U \subseteq \prod_1^2(X)$ definable sets, if U is open and nonempty, and X_a has nonempty interior for every $a \in U$, then X has nonempty interior.

Notice that in the original definition [6, 9.1] there are some additional conditions; but those conditions are trivially satisfied when $\mathbb{K}$ is an ordered field.

Theorem B. *(See Section 6.) Let $\mathbb{K}$ be locally o-minimal. Then, $\mathbb{K}$ is a d-minimal topological structure. Moreover, let A be a proper, dense, elementary substructure of $\mathbb{K}$, and let $\langle \mathbb{K}, A \rangle$ be the expansion of $\mathbb{K}$ with a new predicate for A. Then, $\mathbb{K}$ is the open core of $\langle \mathbb{K}, A \rangle$. Moreover, the theory of $\langle \mathbb{K}, A \rangle$ is uniquely determined by the theory of $\mathbb{K}$.*

As it is clear from Theorem A, the class of structures with locally o-minimal open core is also elementary; a related question about structures with o-minimal open core is still open (see the discussion after Corollary 3.6).

Definably complete structures were explicitly defined and studied in [13]. The open core of $\mathbb{K}$ was defined already in [15], where they study the case when $\mathbb{K}$ is an expansion of $\mathbb{R}$. Structures with o-minimal open core are the main topic of [2]; here, instead, they are only a side remark, because we show that many of the results of [2] can be generalised to structures with locally o-minimal open core; moreover, we answer some questions left open there. Besides, some of the techniques used in [2] will be also employed here (see Section 2.4 and Section 4). Locally o-minimal structures were introduced in [14], and [2] proved some results on definably complete locally o-minimal structures (see Section 3.1). In his recent preprints [19,18], H. Schoutens studies locally o-minimal structures without the assumption that they expand a field (he calls them "DCTC structures"), as a step in the project of studying "o-minimalistic" structures (i.e., structures that are elementarily equivalent to ultraproducts of o-minimal structures: see also [17] for a development in that direction); notice that our motivation is different: we study structures with locally o-minimal open core as a necessary step in the study of definably complete structures. Schoutens also independently obtains some of the results exposed here; in particular, he achieves a "quasi-cell decomposition" for sets definable in locally o-minimal structures, and a kind of cell decomposition for a certain class of locally o-minimal structures, that he calls "tame".

One of the natural examples of structures with o-minimal open core is given by dense elementary pairs of o-minimal structures $A \prec B$, studied in [21]. The main results of [21] can be generalised to elementary pairs of locally o-minimal structures, with a similar proof (Theorem B; see Section 6). Similar results hold in the more general setting of d-minimal structures, which will be studied in another article [4].

While o-minimal structures are geometric (that is, they eliminate the quantifier $\exists^\infty$ and the algebraic closure acl satisfies the Exchange Principle), no such result is true for locally o-minimal structures: more precisely, if $\mathbb{K}$ is a sufficiently saturated locally o-minimal non-o-minimal structure, then $\mathbb{K}$ does *not* eliminate the quantifier $\exists^\infty$ and acl does *not* satisfy the Exchange Principle (this is true not only for locally o-minimal structures, but for d-minimal structures in general: the proof will be given in [4]). However, we still have a notion of dimension for locally o-minimal structures, given by the topology